Question

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, show that:

(i) $a \cdot (a \times b) = 0$ and $b \cdot (a \times b) = 0$;

(ii) $a \cdot (b \times c) = (a \times b) \cdot c$;

(iii) $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$.

Illustrate these results when $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = \mathbf{i} - 4\mathbf{k}$.

Answer

(i) $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \\
&= a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_1a_2b_3 + a_1a_3b_2 - a_2a_3b_1 = 0
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \\
&= a_2b_1b_3 - a_3b_1b_3 + a_3b_1b_2 - a_1b_2b_3 + a_1b_2b_3 - a_2b_1b_3 = 0
\end{aligned}$$

(ii) $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix}.$

Show that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

$$\begin{aligned}
\text{LHS}: \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix} \\
&= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1
\end{aligned}$$

$$\begin{aligned}
\text{RHS}: (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 a_1b_2 - a_2b_1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\
&= a_2b_3c_1 - a_3b_2c_1 + a_3b_1c_2 - a_1b_3c_2 + a_1b_2c_3 - a_2b_1c_3
\end{aligned}$$

Note that

$$LHS = RHS$$

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(iii) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

LHS:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_2 c_3 - b_3 c_2 \\ b_3 c_1 - b_1 c_3 \\ b_1 c_2 - b_2 c_1 \end{pmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_2 c_3 - b_3 c_2 & b_3 c_1 - b_1 c_3 & b_1 c_2 - b_2 c_1 \end{vmatrix}$$

$$= \begin{pmatrix} a_2 b_1 c_2 - a_2 b_2 c_1 - a_3 b_3 c_1 + a_3 b_1 c_3 \\ a_3 b_2 c_3 - a_3 b_3 c_2 - a_1 b_1 c_2 + a_1 b_2 c_1 \\ a_1 b_3 c_1 - a_1 b_1 c_3 - a_2 b_2 c_3 + a_2 b_3 c_2 \end{pmatrix}$$

RHS:

$$(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} = \left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right] \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - \left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right] \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= (a_1 c_1 + a_2 c_2 + a_3 c_3) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - (a_1 b_1 + a_2 b_2 + a_3 b_3) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 b_1 c_1 + a_2 b_1 c_2 + a_3 b_1 c_3 - a_1 b_1 c_1 - a_2 b_2 c_1 - a_3 b_3 c_1 \\ a_1 b_2 c_1 + a_2 b_2 c_2 + a_3 b_2 c_3 - a_1 b_1 c_2 - a_2 b_2 c_2 - a_3 b_3 c_2 \\ a_1 b_3 c_1 b + a_2 b_3 c_2 + a_3 b_3 c_3 - a_1 b_1 c_3 - a_2 b_2 c_3 - a_3 b_3 c_3 \end{pmatrix}$$

Note that
$$LHS = RHS$$

If $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$

then $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix}$

and $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ 1 & 0 & -4 \end{vmatrix} = \begin{pmatrix} 8 \\ 7 \\ 2 \end{pmatrix}$

(i) $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix} = 5 - 2 - 3 = 0.$

$$\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix} = 5 + 4 - 9 = 0.$$

(ii) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 7 \\ 2 \end{pmatrix} = 8 + 7 + 2 = 17.$

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = 5 + 0 + 12 = 17.$$

So $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

(iii) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 8 \\ 7 \\ 2 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 8 & 7 & 2 \end{vmatrix} = \begin{pmatrix} -5 \\ 6 \\ -1 \end{pmatrix}$

$$(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} = \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} \right] \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} - \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$$

$$= (1 + 0 - 4) \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} - (1 - 2 + 3) \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$$

$$= -3 \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 6 \\ -1 \end{pmatrix}$$

So

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$