Question

Suppose $f(z) = p(z)/q(z)$ where $p(z_0) \neq 0,\ \ q(z_0) = 0$ and $q'(z_0) \neq 0$. Show that $f(z)$ has a simple pole at z_0 with residue $p(z_0)/q'(z_0)$.

Evaluate the following integrals. Justify any limits used.

i) $\displaystyle\int_0^\infty \frac{x^2 dx}{(x^2+1)(x^2+4)}$

ii) $\displaystyle\int_0^{2\pi} \frac{d\theta}{9\cos^2\theta + 4\sin^2\theta}$

Answer

$f(z) = \dfrac{p(z)}{q(z)}$

$(z - z_0)f(z) = p(z)\dfrac{(z - z_0)}{q(z) - q(z_0)} \to \dfrac{P(z_0}{q'(z_0)}$ as $z \to z_0$

Thus $f(z)$ has a simple pole at z_0 with residue $\dfrac{P(z_0)}{q'(z_0)}$.

i) We integrate $f(z)\dfrac{z^2}{(z^2+1)(z^2+4)} = \dfrac{z^2}{z^4+5z^2+4} = \dfrac{p(z)}{q(z)}$

The contour used is $C = \{-R, R\} + \{Re^{it}\ \ 0 \leq t \leq \pi\}$. Within this the singularities are at $z = i$ and $z = 2i$.

We find the residues:

$z = i \qquad \text{res} = \left.\dfrac{z^2}{4z^3 + 10z}\right|_{z=i} = \dfrac{-1}{-4i + 10i} = \dfrac{-1}{6i} = \dfrac{i}{6}$

$z = 2i \qquad \text{res} = \left.\dfrac{z^2}{4z^3 + 10z}\right|_{z=2i} = \dfrac{-4}{-32i + 20i} = \dfrac{1}{3i} = \dfrac{-i}{3}$

So, provided $R > 2$, $\displaystyle\int_C f(z)dz = 2\pi i\left(\frac{-i}{6}\right) = \frac{\pi}{3}$

Now for $|z| = R$, $|f(z)| = \dfrac{R^2}{|z^2+1||z^2+4|} \leq \dfrac{R^2}{(R^2-1)(R^2-4)}$

So for the semi-circular part C_1 of the contour,

$\displaystyle\int_{C_1} f(z)dz \leq \frac{R^2\pi R}{(R^2-1)(R^2-4)} \to 0$ as $R \to \infty$.

Thus, letting $R \to \infty$, we obtain $\displaystyle\int_{-\infty}^\infty f(x)dx = \frac{\pi}{3}$, so $\displaystyle\int_0^\infty f(x)dx = \frac{\pi}{6}$

ii) Let $z = e^{i\theta}$. So $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$, $\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$, $d\theta = \frac{dz}{iz}$

So the integral becomes

$$\int_C \frac{dz}{iz\left(\frac{9}{4}\left(z^2 + 2 + \frac{1}{z^2}\right) - \frac{4}{4}\left(z^2 - 2 + \frac{1}{z^2}\right)\right)} \qquad (C\text{=unit circle})$$

$$= \frac{4}{i}\int_C \frac{zdz}{5z^4 + 26z^2 + 5} = \frac{4}{i}\int_C \frac{zdz}{(5z^2+1)(z^2+5)}$$

The integrand has simple poles inside C at $z = \pm\frac{i}{\sqrt{5}}$

$$\text{Res}\left(\frac{+i}{\sqrt{5}}\right) = \left.\frac{z}{20z^3 + 52z}\right|_{\pm\frac{i}{\sqrt{5}}} = \frac{1}{-4+52} = \frac{1}{48}$$

So the integral is $\frac{4}{i}2\pi i\left(\frac{1}{48} + \frac{1}{48}\right) = \frac{\pi}{3}$