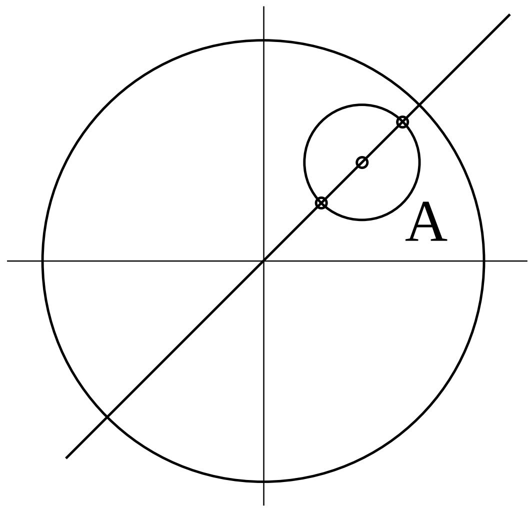
Question

Let A be the Euclidean circle in $\mathbf{D}$ with Euclidean center $\frac{1}{3}+\frac{1}{3}i$ and Euclidean radius $\frac{1}{10}$. Determine the hyperbolic center and hyperbolic radius of A.

Answer



The diameter of A is the euclidean/hyperbolic line through 0 and $c = \dfrac{1}{3}(1+i) = \dfrac{\sqrt{2}}{3}e^{i\frac{\pi}{4}}$ and so we need to consider the two points of intersection of this line and A

$$p = \left(\frac{\sqrt{2}}{3}+\frac{1}{10}\right)e^{i\frac{\pi}{4}} \text{ and } q = \left(\frac{\sqrt{2}}{3}-\frac{1}{10}\right)e^{i\frac{\pi}{4}}$$

$$\begin{aligned} d_{\mathbf{D}}(p,q) &= d_{\mathbf{D}}\left(\left(\frac{\sqrt{2}}{3}+\frac{1}{10}\right)e^{i\frac{\pi}{4}}, \left(\frac{\sqrt{2}}{3}-\frac{1}{10}\right)e^{i\frac{\pi}{4}}\right) \\ &= d_{\mathbf{D}}\left(\frac{\sqrt{2}}{3}+\frac{1}{10}, \frac{\sqrt{2}}{3}-\frac{1}{10}\right) \quad \text{since } m(z) = e^{\frac{-i\pi}{4}}z \text{ lies in } Isom(\mathbf{D}, d_{\mathbf{D}}) \\ &= d_{\mathbf{D}}\left(\frac{\sqrt{2}}{3}+\frac{1}{10}, 0\right) - d_{\mathbf{D}}\left(\frac{\sqrt{2}}{3}-\frac{1}{10}, 0\right) \end{aligned}$$

(since $\dfrac{\sqrt{2}}{3}+\dfrac{1}{10}, \dfrac{\sqrt{2}}{3}-\dfrac{1}{10}, 0$ all lie on a hyperbolic line)

$$d_{\mathbf{D}}\left(\frac{\sqrt{2}}{3}+\frac{1}{10}, 0\right) = \ln\left(\frac{1+\frac{\sqrt{2}}{3}+\frac{1}{10}}{1-\frac{\sqrt{2}}{3}-\frac{1}{10}}\right) = \ln\left(\frac{33+10\sqrt{2}}{27-10\sqrt{2}}\right)$$

$$d_{\mathbf{D}}\left(\frac{\sqrt{2}}{3}-\frac{1}{10}, 0\right) = \ln\left(\frac{1+\frac{\sqrt{2}}{3}-\frac{1}{10}}{1-\frac{\sqrt{2}}{3}+\frac{1}{10}}\right) = \ln\left(\frac{27+10\sqrt{2}}{33-10\sqrt{2}}\right)$$

So,

$$\begin{aligned} d_{\mathbf{D}}\left(\frac{\sqrt{2}}{3}+\frac{1}{10},\frac{\sqrt{2}}{3}-\frac{1}{10}\right) &= \ln\left(\frac{33+10\sqrt{2}}{27-10\sqrt{2}}\cdot\frac{33-10\sqrt{2}}{27+10\sqrt{2}}\right) \\ &= \ln\left(\frac{1089-200}{729-200}\right) \\ &= \ln\left(\frac{889}{529}\right) \end{aligned}$$

So, the hyperbolic radius of A is $\frac{1}{2}\ln\left(\frac{889}{529}\right)$.

The hyperbolic center is the point $ce^{\frac{i\pi}{4}}$, where $\frac{\sqrt{2}}{3}-\frac{1}{10}<c<\frac{\sqrt{2}}{3}+\frac{1}{10}$ and

$\underline{d_{\mathbf{D}}\left(\frac{\sqrt{2}}{3}-\frac{1}{10},c\right)=\frac{1}{2}\ln\left(\frac{889}{529}\right)=0.25955}$

$$d_{\mathbf{D}}(0,c)-d_{\mathbf{D}}\left(0,\frac{\sqrt{2}}{3}-\frac{1}{10}\right)=\frac{1}{2}\ln\left(\frac{889}{529}\right)$$

$$d_{\mathbf{D}}(0,c)=\frac{1}{2}\ln\left(\frac{889}{529}\right)+\ln\left(\frac{27+10\sqrt{2}}{33-10\sqrt{2}}\right)=\alpha=1.039$$

$\underline{c=\tanh\left(\frac{1}{2}\alpha\right)}$

$$=\frac{e^{\frac{1}{2}\alpha}-e^{-\frac{1}{2}\alpha}}{e^{\frac{1}{2}\alpha}+e^{\frac{1}{2}\alpha}}=\frac{e^{\alpha}-1}{e^{\alpha}+1}=0.4775$$