Question

Consider the map $\varphi : \mathbf{D} \to \mathbf{D}$ given by $\varphi(z) = z^2$. Calculate the pullback of the hyperbolic element of arc-length from $\mathbf{D}$ using φ. (That is, define a new element of arc-length on $\mathbf{D}$ by setting $\text{length}(f) = \text{length}_{\mathbf{D}}(\varphi \circ f)$ for a path $f : [a, b] \to \mathbf{D}$.) Is the pullback of the hyperbolic element of arc-length by φ the hyperbolic element of arc-length on $\mathbf{D}$?

Answer

The pull back of the standard hyperbolic metric on $\mathbf{D}$:

$$\phi : \mathbf{D} \longrightarrow \mathbf{D} \quad f : [a, b] \longrightarrow \mathbf{D} \text{ a path}$$

$$\begin{aligned}
\text{length}(f) &= \text{length}_{\mathbf{D}}(\phi \circ f) \\
&= \int_{\phi \circ f} \frac{1}{1 - |z|^2} |dz| \\
&= \int_a^b \frac{2}{1 - |\phi(f(t))|^2} |\phi'(f(t))||f'(t)|\, dt \\
&= \int_f \frac{2}{1 - |\phi(z)|^2} |\phi'(z)||\, dz|
\end{aligned}$$

with $\phi(z) = z^2$:

$$\begin{aligned}\frac{2}{1-|\phi(z)|^2} &= \frac{2}{1-|z|^4} 2|z| \\ &= \frac{4|z|}{1-|z|^4}\end{aligned}$$

Since $\dfrac{2}{1-|z|^2} = \dfrac{4|z|}{1-|z|^4}$, we do not recover the hyperbolic metric on $\mathbf{D}$ via the pull back.