Question

Calculate the hyperbolic distance in $\mathbf{H}$ between $z = 5i$ and $w = e^{i\theta}$, where $0 < \theta < \pi$. For what value (or values) of θ (if any) is this distance minimized?

Answer

$p = 5i, q = e^{i\theta} = \cos(\theta) + i\sin(\theta)$

The euclidean line segment joining p and q has midpoint $\frac{1}{2}\cos(\theta) + i\frac{1}{2}(5 + \sin(\theta))$ and slope $m = \frac{\sin(\theta) - 5}{\cos(\theta)}$ and so the perpendicular bisector has equation

$$y - \frac{1}{2}(5 + \sin(\theta)0 = \frac{\cos(\theta)}{5 - \sin(\theta)}(x - \frac{1}{2}\cos(\theta))$$

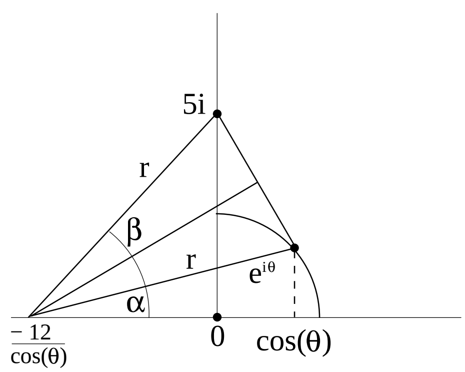
Setting $y = 0$:

$$\frac{\sin(\theta) - 5}{2\cos(\theta)}(5 + \sin(\theta)) = x - \frac{1}{2}\cos(\theta)$$

$$\frac{\sin^2(\theta) - 25}{2\cos(\theta)} + \frac{1}{2}\cos(\theta) = x \quad \text{so } x = \frac{-12}{\cos(\theta)}$$

The center of the euclidean circle containing the line through p and q is $c = \frac{-12}{\cos(\theta)}$ and the radius is $r = |c - 5i|$.

Suppose now that $0 < \theta \leq \frac{\pi}{2}$ (if $\theta > \frac{\pi}{2}$, then we use that $d_{\mathbf{H}}(e^{i\theta}, 5i) = d_{\mathbf{H}}(B(e^{i\theta}), B(5i)) = d_{\mathbf{H}}(e^{i(\pi-\theta)}, 5i)$ and $0 < \pi - \theta \leq \frac{\pi}{2}$).



$$sin(\alpha) = \frac{\sin(\theta)}{r}$$
$$sin(\beta) = \frac{5}{r}$$

Parametrize the hyperbolic line segment between $5i$ and $e^{i\theta}$ by $f(t) = re^{it} + c,\ \alpha \leq t \leq \beta$. $(0 < \frac{\pi}{2})$

$$
\begin{aligned}
d_{\mathbf{H}}(e^{i\theta}, 5i) &= \text{length}_{\mathbf{H}}(\text{f}) \\
&= \int_\alpha^\beta \frac{1}{\sin(s)}\, ds \\
&= \ln\left|\frac{\csc(\beta)-\cot(\beta)}{\csc(\alpha)-\cot(\alpha)}\right|
\end{aligned}
$$

$$
\cos(\alpha) = \frac{\frac{12}{\cos(\theta)+\cos(\theta)}}{r} \quad \sin(\alpha) = \frac{\sin(\theta)}{r}
$$

$$
\cos(\beta) = \frac{12}{\cos(\theta) r} \quad \sin(\beta) = \frac{5}{r}
$$

$$
\csc(\beta) - \cot(\beta) = \frac{r}{5} - \frac{12}{5\cos(\theta)} = \frac{r\cos(\theta) - 12}{5\cos(\theta)}
$$

$$
\csc(\alpha) - \cot(\alpha) = \frac{r}{\sin(\theta)} - \frac{\frac{12}{\cos(\theta)} + \cos(\theta)}{\sin(\theta)} = \frac{r\cos(\theta) - 12 - \cos^2(\theta)}{\sin(\theta)\cos(\theta)}
$$

$$
\begin{aligned}
\frac{\csc(\beta)-\cot(\beta)}{\csc(\alpha)-\cot(\alpha)} &= \frac{(r\cos(\theta 0 - 12)\sin(\theta)}{5(r\cos(\theta) - 12 - \cos^2(\theta))} \\
r &= \sqrt{\frac{144}{\cos^2(\theta)} + 25} \\
r\cos(\theta) &= \sqrt{144 + 25\cos^2(\theta)} \quad (\text{since } \cos(\theta) > 0).
\end{aligned}
$$

$$
\frac{\csc(\beta)-\cot(\beta)}{\csc(\alpha)-\cot(\alpha)} = \frac{(\sqrt{144 + 25\cos^2(\theta)} - 12)\sin(\theta)}{5(\sqrt{144 + 25\cos^2(\theta)} - 12 - \cos^2(\theta))}
$$