QUESTION The time to failure,t in hours, of components follows the density function $f(t) = \alpha^2 te{-}\alpha t, t > 0$.

(a) What is the probability that a component which has survived for two hours will fail in the next hour?

(b) The cost of producing a component is $2\mu^2$ units where μ is the mean time for failure. An income of 48t units per component is received for the time the component is working properly. Show that the maximum expected profit per component is 288 units corresponding to $\alpha = \frac{1}{6}$.

ANSWER $f(t) = \alpha^2 te^{-\alpha t},\ t > o$

$$\begin{aligned}
F(t) &= \int_0^t \alpha^2 se^{-\alpha s}\, ds \\
&= [-\alpha se^{-\alpha}s]_o^t + \int_0^t \alpha e^{-\alpha s}\, ds \\
&= -\alpha te^{-\alpha t} - [e^{-\alpha s}]_0^t \\
&= 1 - e^{-\alpha t}(1 + \alpha t)
\end{aligned}$$

(a)

$$\begin{aligned}
\text{Require} P(2, T, 3|t > 2) &= \frac{P(2 < T < 3 \text{ and } T > 2}{P(T > 2)} \\
&= \frac{P(2 < T < 3)}{P(T > 2)} = \frac{F(3) - F(2)}{1 - F(2)} \\
&= \frac{1 - e^{-3\alpha}(1 + 3\alpha) - (1 - e^{-2\alpha}(1 + 2\alpha))}{e^{-2\alpha}(1 + 2\alpha)} \\
&= e^{-2\alpha}(1 + 2\alpha) - e^{-3\alpha}(1 + 3\alpha)e^{-2\alpha}(1 + 2\alpha) \\
&= 1 + \frac{e^{-\alpha}(1 + 3\alpha)}{1 + 2\alpha}
\end{aligned}$$

(b)

$$\begin{aligned}
\mu &= \int_0^\infty \alpha^2 t^2 e^{\alpha t}\, dt \\
&= [-\alpha e^{-\alpha t} t^2]_0^\infty + \int_0^\infty 2t\alpha e^{-\alpha t}\, dt \\
&= 0 + \frac{2}{\alpha}\int_0^\infty \alpha^2 te^{-\alpha t}\, dt \\
&= \frac{2}{\alpha} \quad (\text{since} \int f(t)\, dt = 1)
\end{aligned}$$

Profit$= 48T - 2\mu^2$
E(Profit)$= 48E(T) - \mu^2 = 48\mu - 2\mu^2$
$\frac{dE(\text{Profit})}{d\mu} = 48 - 4\mu = 0$ when $\mu = 12$
$\frac{d^2E(\text{Profit})}{d\mu^2} = -4$ therefore profit is maximum when $\mu = 12$. Hence $\frac{2}{\alpha} = 12$ therefore $\alpha = \frac{1}{6}$