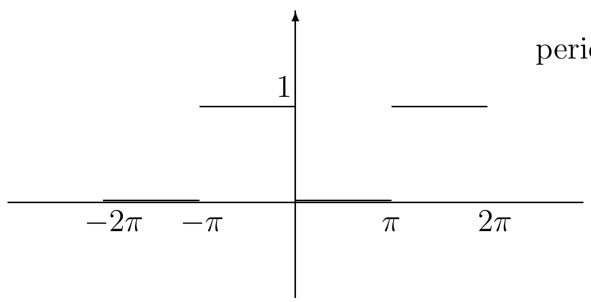
QUESTION

(b) Using the formula sheet derive the Fourier series for the periodic function $f(t)$ which is defined by

$$f(t) = \begin{cases} 1 & -\pi \leq t < 0, \\ 0 & 0 \leq t < \pi, \end{cases} \quad \text{and } f(t+2\pi) = f(t) \text{ for all } t.$$

ANSWER

(b)



period $= 2\pi = T$

Fourier series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(nt) + b_n \sin(nt))$.

$$\begin{aligned} a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt &= \frac{1}{\pi}\int_{-\pi}^{0} 1\,dt + \frac{1}{\pi}\int_{0}^{\pi} 0\,dt \\ &= \frac{1}{\pi}[t]_{-\pi}^{0} = \frac{1}{\pi}(0 - (-\pi)) = 1 \end{aligned}$$

$$\begin{aligned} a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt &= \frac{1}{\pi}\int_{-\pi}^{0} \cos(nt)\,dt + \frac{1}{\pi}\int_{0}^{\pi} 0\,dt \\ &= \frac{1}{\pi}\left[\frac{\sin(nt)}{n}\right]_{-\pi}^{0} \\ &= \frac{1}{\pi n}(\sin(0) - \sin(-n\pi)) = 0 \end{aligned}$$

$$\begin{aligned} b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt &= \frac{1}{\pi}\int_{-\pi}^{0} \sin(nt)\,dt + \frac{1}{\pi}\int_{0}^{\pi} 0\,dt \\ &= \frac{1}{\pi}\left[-\frac{\cos(nt)}{n}\right]_{-\pi}^{0} \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{\pi n}[\cos(0) - \cos(-n\pi)] \\
&= -\frac{1}{\pi n}[1 - \cos(n\pi)] \\
&= -\frac{1}{\pi n}[1 - (-1)^n] \\
&= \begin{cases} 0 & n \text{ even} \\ -\frac{2}{\pi n} & n \text{ odd} \end{cases}
\end{aligned}$$

Therefore the Fourier series is

$$\frac{1}{2} - \frac{2}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin(nt) = \frac{1}{2} - \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{1}{(2m+1)} \sin((2m+1)t)$$