QUESTION

(a) If f is a function of u and v, where $u = \dfrac{x}{y}$ and $v = xy$, use the chain rule to show that

$$\frac{\partial f}{\partial x} = \frac{1}{y}\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v},$$

and find the corresponding expression for $\dfrac{\partial^2 f}{\partial x^2}$.

(b) State the general form of Maclaurin's theorem with a remainder. Use this to show

$$e^{2x} = 1 + 2x + 2x^2 + R_2.$$

State the Lagrange form of the remainder R_2, and determine its maximum value when $0 \le x \le 0.25$.

ANSWER

(a) Using the chain rule $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x}$.

$\dfrac{\partial u}{\partial x} = \dfrac{1}{y}$, $\dfrac{\partial v}{\partial x} = y$, and hence

$\dfrac{\partial f}{\partial x} = \dfrac{1}{y}\dfrac{\partial f}{\partial u} + y\dfrac{\partial f}{\partial v}$

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{1}{y}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial x}\left(y\frac{\partial f}{\partial v}\right) \\
&= \frac{1}{y}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\right) + y\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\right) \\
&= \frac{1}{y}\left\{\frac{\partial^2 f}{\partial u^2}\underbrace{\frac{\partial u}{\partial x}}_{=\frac{1}{y}} + \frac{\partial^2 f}{\partial v\partial u}\underbrace{\frac{\partial v}{\partial x}}_{=y}\right\} + y\left\{\frac{\partial^2 f}{\partial u\partial v}\underbrace{\frac{\partial u}{\partial x}}_{=\frac{1}{y}} + \frac{\partial^2 f}{\partial v^2}\underbrace{\frac{\partial v}{\partial x}}_{=y}\right\} \\
&= \frac{1}{y^2}\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v\partial u} + \frac{\partial^2 f}{\partial u\partial v} + y^2\frac{\partial^2 f}{\partial v^2} \\
&= \frac{1}{y^2}\frac{\partial^2 f}{\partial u^2} + 2\frac{\partial^2 f}{\partial u\partial v} + y^2\frac{\partial^2 f}{\partial v^2}
\end{aligned}$$

(b) Maclaurin's theorem:

$$f(x) = f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + R_n$$

where $R_n = \dfrac{x^{n+1}}{(n+1)!}f^{n+1}(\theta x),\ 0 < \theta < 1$

$$\begin{array}{ll} f(x) = e^{2x} & f(0) = e^0 = 1 \\ f^{(1)}(x) = 2e^{2x} & f^{(1)}(0) = 2e^0 = 2 \\ f^{(2)}(x) = 4e^{2x} & f^{(2)}(0) = 4e^0 = 4 \\ f^{(3)}(x) = 8e^{2x} & \end{array}$$

Substituting these values into Maclaurin's theorem,

$$e^{2x} = 1 + x(2) + \frac{x^2}{2!}(4) + R_2 = 1 + 2x + 2x^2 + R_2,\ R_2 = \frac{x^3}{3!}f^{(3)}(\theta x)$$

$f^{(3)}(\theta x) = 8e^{2\theta x} < 8e^{\frac{1}{2}}$, since $0 \leq 2x \leq 0.5$ and $0 < \theta < 1$

Hence $|R_2| < 8e^{\frac{1}{2}}\left(\frac{x^3}{6}\right) < \frac{4e^{\frac{1}{2}}}{3}\left(\frac{1}{4}\right)^3 = \frac{e^{\frac{1}{2}}}{48} = 0.0343$