QUESTION

(i) Evaluate the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 2 & 5 & \beta \end{pmatrix},$$

where β is a constant.

(ii) For what value of β does $\mathbf{A}^{-1}$ not exist?

(iii) Determine $\mathbf{A}^{-1}$ when $\beta = 3$, and verify that your answer satisfies the equations $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

ANSWER

(i) $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 2 & 5 & \beta \end{pmatrix},$

so $\det \mathbf{A} = 1(3\beta - (-1)5) - 2(2\beta - (-1)2) + 0 = 3\beta + 5 - 4\beta - 4 = 1 - \beta,$

(ii) $\mathbf{A}^{-1}$ does not exist if $\det\mathbf{A} = 0$, i.e.$\beta = 1$

(iii) Using Gaussian elimination

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 2 & 5 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 3 & -2 & 0 & 1 \end{array}\right), \ r_2' = r_2 - 2r_1, \ r_3' = r_3 - 2r_1$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & -2 & 0 & 1 \end{array}\right), \ r_2' = -r_2$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -3 & 2 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 2 & -4 & 1 & 1 \end{array}\right), \ r_1' = r_1 - 2r_2, \ r_3' = r_3 - r_2$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -3 & 2 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & \frac{1}{2} \end{array}\right),\ r_3' = \frac{1}{2}r_3$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 3 & 1 \\ 0 & 1 & 0 & 4 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -2 & \frac{1}{2} & \frac{1}{2} \end{array}\right),\ r_1' = r_1 + 2r_3,\ r_2' = r_2 - r_3$$

Hence $\mathbf{A}^{-1} = \begin{pmatrix} -7 & 3 & 1 \\ 4 & -\frac{3}{2} & -\frac{1}{2} \\ -2 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

To verify that this is correct

$$\begin{aligned}
\mathbf{AA^{-1}} &= \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 2 & 5 & 3 \end{pmatrix} \begin{pmatrix} -7 & 3 & 1 \\ 4 & -\frac{3}{2} & -\frac{1}{2} \\ -2 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} -7+8+0 & 3-3+0 & 1-1+0 \\ -14+12+2 & 6-\frac{9}{2}-\frac{1}{2} & 2-\frac{3}{2}-\frac{1}{2} \\ -14+20-6 & 6-\frac{15}{2}+\frac{3}{2} & 2-\frac{5}{2}+\frac{3}{2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\mathbf{A^{-1}A} &= \begin{pmatrix} -7 & 3 & 1 \\ 4 & -\frac{3}{2} & -\frac{1}{2} \\ -2 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 2 & 5 & 3 \end{pmatrix} \\
&= \begin{pmatrix} -7+6+2 & -14+9+5 & 0-3+3 \\ 4-3-1 & 8-\frac{9}{2}-\frac{5}{2} & 0+\frac{3}{2}-\frac{3}{2} \\ -2+1+1 & -4+\frac{3}{2}+\frac{5}{2} & 0-\frac{1}{2}+\frac{3}{2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$