QUESTION

Find the eigenvalues of the matrix

$$\begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

and determine the corresponding eigenvectors.

Verify that the eigenvectors are perpendicular to each other.

ANSWER

$\mathbf{A} = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ The eigenvalues satisfy $\det(A - \lambda I) = 0$

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 1-\lambda & 3 & 0 \\ 3 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} \\ &= (2-\lambda)\{(1-\lambda)^2 - 3^2\} = (2-\lambda)\{(1-\lambda-3)(1-\lambda+3)\} \\ &= (2-\lambda)(-\lambda-2)(4-\lambda) = (2-\lambda)(\lambda+2)(\lambda-4) \\ &= 0 \text{ if } \lambda = 2, -2, 4 \end{aligned}$$

$\lambda = 4:$

$(A - 4I)\mathbf{X} = 0$

$$\begin{pmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} -3x_1 + 3x_2 &= 0 \\ 3x_1 - 3x_2 &= 0 \\ -2x_3 &= 0 \end{aligned}\right\} x_1 = x_2 = C,\ x_3 = 0$$

Therefore the eigenvector is $C \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = -2:$

$(A - (-2)I)\mathbf{X} = 0$

$$\begin{pmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 3x_1 + 3x_2 &= 0 \\ 3x_1 + 3x_2 &= 0 \\ 4x_3 &= 0 \end{aligned}\right\} x_2 = -x_1,\ x_3 = 0$$

Therefore the eigenvector is $D\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\lambda = 2:$

$(A - 2I)\mathbf{X} = 0$

$$\begin{pmatrix} -1 & 3 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} -x_1 + 3x_2 &= 0 \\ 3x_1 - x_2 &= 0 \\ 0 &= 0 \end{aligned}\right\} \rightarrow -x_1 + 3x_2 = 0,\ 8x_2 = 0, \rightarrow x_1 = x_2 = 0,\ x_3 = \text{E}$$

Therefore the eigenvector is $E\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$C(1,1,0) \cdot D(1,-1,0) = CD(1 - 1 + 0) = 0$

$C(1,1,0) \cdot E(0,0,1) = CE(0 + 0 + 0) = 0$

$D(1,-1,0) \cdot E(0,0,1) = DE(0 + 0 + 0) = 0$

So the eigenvectors are perpendicular to each other.