QUESTION

(a) Using the substitution $x = yt$ find the solution of the differential equation

$$t^2\frac{dx}{dt} = x^2 + xt$$

which satisfies the condition $x = 1$ when $t = 1$.

(b) Find the general solution of the second order differential equation

$$\frac{d^2x}{dt^2} - 9x = \cos(4t).$$

ANSWER

(a) $t^2\dfrac{dx}{dt} = x^2 + xt$

Making the substitution $x = yt$, $\dfrac{dx}{dt} = y + t\dfrac{dy}{dt}$, and substituting into the ODE gives

$t^2\left(y + t\dfrac{dy}{dt}\right) = t^2y^2 + (yt)t = t^2y^2 + t^2y.$

Hence $y+t\dfrac{dy}{dt} = y^2+y$, therefore $\dfrac{dy}{dt} = \dfrac{y^2}{t}$ which is a separable equation.

$\displaystyle\int \frac{dy}{y^2} = \int \frac{dt}{t}$ leads to $-\dfrac{1}{y} = \ln t + c \Rightarrow y = -\dfrac{1}{\ln t + c}$

When $t = 1$, $y = 1$ therefore $1 = -\dfrac{1}{\ln 1 + c} = -\dfrac{1}{c}$, so $c = -1$

Therefore $y = -\dfrac{1}{\ln t - 1} = \dfrac{1}{1 - \ln t}$ so $x = \dfrac{t}{1 - \ln t}$.

(b) $\dfrac{d^2x}{dt^2} - 9x = \cos(4t)$

To find the complementary function consider the equation

$\dfrac{d^2x}{dt^2} - 9x = 0.$

This has the auxiliary equation $m^2 - 9 = 0$, $m^2 = 9$, $m = \pm 3$, and therefore the complementary function is $x = Ae^{3t} + Be^{-3t}$

To find a particular integral try $x = C\cos(4t) + D\sin(4t)$,

$\dfrac{dx}{dt} = -4C\sin(4t) + 4D\cos(4t)$,

$\dfrac{d^2x}{dt^2} = -16C\cos(4t) - 16D\sin(4t)$

Substituting this into the ODE gives
$-16C\cos(4t) - 16D\sin(4t) - 9(C\cos(4t) + D\sin(4t)) = \cos(4t)$
i.e. $-25C\cos(4t) - 25D\sin(4t) = \cos(4t)$
Thus $-25C = 1, -25D = 0 \to C = -\frac{1}{25},\ D = 0$
Hence a particular integral is $x = -\frac{1}{25}\cos(4t)$ and the general solution can be written $x = Ae^{3t} + Be^{-3t} - \frac{1}{25}\cos(4t)$