QUESTION Find the solution of the differential equation

$$y'' + 2y' + 2y = 2x + 1 + e^x \cos x$$

which satisfies the boundary conditions $y = y' = 2$ at $x = 0$.

(a) by using particular integral and complementary function,

(b) by using Laplace transforms.

ANSWER $y'' + 2y' + 2y = 2x + 1 + e^x \cos x, \quad y(0) = 2, \quad y'(0) = 2$

(a) $m^2 + 2m + 2 = 0,\ (m+1)^2 = -1,\ m_1 = -1 + i,\ m_2 = -1 - i$
$y_{CF} = C_n e^{-x} \cos x + c_2 e^{-x} \sin x$
$y'' + 2y' + 2y = 2x + 1$ try $y = Ax + B$
$2A + 2(Ax + B) = 2x + 1 \Rightarrow A = 1,\ B = -\frac{1}{2}$
$y'' + 2y' + 2y = e^x \cos x = \mathrm{Re} e^{(1+i)x}$
Try $y = p(x) e^{(1+i)x}$
$(D + 1 - i)(D + 1 + i) p(x) e^{(1+i)x} = e^{(1+i)x}$
$\Rightarrow (D + 2)(D + 2 + 2i) p(x) = 1$
$p(x) = \frac{1}{D+2} \frac{1}{D+2+2i} = \frac{1}{4(1+i)} = \frac{1-i}{8}$
$y = \mathrm{Re} \frac{1-i}{8} e^{(1+i)x} = \frac{1}{8} e^x (\cos x + \sin x)$
General solution:

$$\begin{aligned}
Y &= e^{-x}(C_1 \cos x + C_2 \sin x) + \frac{1}{8} e^x (\cos x + \sin x) + x - \frac{1}{2} \\
y(0) &= C_1 + \frac{1}{8} - \frac{1}{2} = 2 \Rightarrow C_1 = \frac{16 + 4 - 1}{8} = \frac{19}{8} \\
y'(0) &= C_2 - C_1 + \frac{1}{8}(1 + 1) + 1 = 2 \Rightarrow C_2 = \frac{19 - 2 - 8 + 16}{8} = \frac{25}{8}
\end{aligned}$$

(b) $(s^2 Y(s) - s y(0) - y'(0)) + 2(sY(s) - y(0)) + 2Y(s) = \mathcal{L}(2x + 1 + e^x \cos x)$
$(s^2 + 2s + 2) Y(3) = 2s + 6 + \frac{2}{s^2} + \frac{1}{s} + \frac{s-1}{(s-1)^2 + 1}$
Use complex inversion formula ...