Question

(a) Reduce the conic

$$6x^2 - 24xy - y^2 + 60x - 20y + 45 = 0$$

to standard form, and identify its type.

(b) Find the eccentricity, foci, and semi latus rectum of the ellipse

$$3x^2 + 4y^2 = 1$$

Hence write down all its polar equation referred to a focus as origin and major axis as initial line.

Answer

(a) $6x^2 - 24xy - y^2 + 60x - 20y + 45 = 0$

Let $x = \xi + h$ and $y = \eta + k$

$6(\xi + h)^2 - 24(\xi + h)(\eta + k) - (\eta + k)^2 + 60(\xi + h) - 20(\eta + k) + 45 = 0$

$6\xi^2 - 24\xi\eta - \eta^2 + \xi(12h - 24k + 60) - \eta(24h + 2k + 20) + 6h^2 - 24hk - k2 + 60h - 29k + 45 = 0$

Choose h, k so that

$$\left.\begin{array}{c} 12h - 24k + 60 \\ 24h + 2k + 20 \end{array}\right\} k = 2, h = -1$$

So the equation becomes

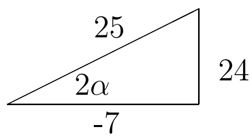
$$6\xi^2 - 24\xi\eta - \eta^2 = 5$$

Now $\xi = X\cos\alpha - Y\sin\alpha$ and $\eta = X\sin\alpha + Y\cos\alpha$ gives

$$\begin{aligned} 5 &= 6(X^2\cos^2\alpha - 2XY\cos\alpha\sin\alpha + Y^2\sin^2\alpha \\ &\quad -24(X\cos\alpha - Y\sin\alpha)(X\sin\alpha + Y\cos\alpha) \\ &\quad -(X^2\sin^2\alpha + 2XY\cos\alpha\sin\alpha + Y^2\cos^2\alpha) \end{aligned}$$

$$\begin{aligned} 5 &= X^2(6\cos^2\alpha - 24\cos\alpha\sin\alpha - \sin^2\alpha) \\ &\quad Y^2(6\sin^2\alpha + 24\cos\alpha\sin\alpha - \cos^2\alpha) \\ &\quad XY(-12\cos\alpha\sin\alpha - 24(\cos^2\alpha - \sin^2\alpha) - 2(\cos\alpha\sin\alpha) \end{aligned}$$

We want the coefficient of XY to be zero $-24\cos 2\alpha - 7\sin 2\alpha = 0$ So $\tan 2\alpha = -\frac{24}{7}$



Choose $\cos 2\alpha = -\frac{7}{25}$ and $\sin \alpha = \frac{24}{25}$
$\cos^2\alpha = \frac{1}{2}(\cos 2\alpha + 1) = \frac{9}{25}$ $\quad \sin^2\alpha = \frac{16}{25}$

$$3Y^2 - 2X^2 = 1$$

(b) Eccentricity $= \dfrac{1}{2}$

Foci $= \left(\pm\dfrac{1}{2\sqrt{3}}, 0\right)$

$l = \dfrac{\sqrt{3}}{4}$

Polar equation $\dfrac{\sqrt{3}}{4r} = 1 \pm \dfrac{1}{2}\cos\theta$