Question

Show that as $x \to \infty$

$$\int_x^\infty e^{-t}t^{\lambda-1}\,dt \sim x^\lambda e^{-x}\left[\frac{1}{x}+\frac{\lambda-1}{x}+\frac{(\lambda-1)(\lambda-2)}{x^2}+O\left(\frac{1}{x^4}\right)\right]$$

Answer

Use integration by parts

$I = \displaystyle\int_x^\infty e^{-t}t^{\lambda-1}\,dt$

$$\begin{array}{rclrcl} u & = & t^{\lambda-1} & dv & = & e^{-t} \\ du & = & (\lambda-1)t^{\lambda-1} & v & = & -e^{-t} \end{array}$$

$$\begin{aligned}
I &= [-t^{\lambda-1}e^{-t}]_x^\infty + (\lambda-1)\int_x^\infty t^{\lambda-2e^{-t}}\,dt \\
&= x^{\lambda-1}e^{-x} + (\lambda-1)\left[[-t^{\lambda-1}e^{-t}]_x^\infty + \int_x^\infty (\lambda-2)t^{\lambda-3}e^{-t}\,dt\right]
\end{aligned}$$

$$\begin{array}{rclrcl} u & = & t^{\lambda-2} & dv & = & e^{-t} \\ du & = & (\lambda-2)t^{\lambda-3} & v & = & -e^{-t} \end{array}$$

$$= x^{\lambda-1}e^{-x} + (\lambda-1)x^{\lambda-2}e^{-x} + (\lambda-1)(\lambda-2)\int_x^\infty e^{-t}t^{\lambda-3}\,dt$$

$$\begin{array}{rclrcl} u & = & t^{\lambda-3} & dv & = & e^{-t} \\ du & = & (\lambda-3)t^{\lambda-4} & v & = & -e^{-t} \end{array}$$

$$\vdots$$

$$\begin{aligned}
&= x^{\lambda-1}e^{-x} + (\lambda-1)x^{\lambda-2}e^{-x} + (\lambda-1)(\lambda-2)(\lambda-3)x^{\lambda-3}e^{-x} \\
&\quad +(\lambda-1)(\lambda-2)(\lambda-3)\int_x^\infty t^{\lambda-4}e^{-t}\,dt \\
&= x^\lambda e^{-x}\left(\frac{1}{x}+\frac{(\lambda-1)}{x^2}+\frac{(\lambda-1)(\lambda-2)(\lambda-3)}{x^3}+\frac{R}{x^\lambda e^{-x}}\right)
\end{aligned}$$

$$\begin{aligned}
R &= (\lambda-1)(\lambda-2)(\lambda-3)\int_x^\infty t^{\lambda-4}e^{-t} \\
&\le |(\lambda-1)(\lambda-2)(\lambda-3)|\left|\int_x^\infty t^{\lambda-4}e^{-t}\,dt\right| \\
&\le (\lambda-1)(\lambda-2)(\lambda-3)\int_x^\infty |t^{\lambda-4}|e^{-t}\,dt \\
&\quad x > 0,\ \text{so} \\
&\le (\lambda-1)(\lambda-2)(\lambda-3)\int_x^\infty t^{\lambda-4}e^{-t}\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(\lambda-1)(\lambda-2)(\lambda-3)}{x^{4-\lambda}} \int_x^\infty t^{\lambda-4} x^{4-\lambda} e^{-t}\, dt \\
&= \frac{(\lambda-1)(\lambda-2)(\lambda-3)}{x^{4-\lambda}} \int_x^\infty \left(\frac{x}{t}\right)^{4-\lambda} e^{-t}\, dt \\
&\text{But t} \geq \text{x. Therefore } \left(\frac{\text{x}}{\text{t}}\right) \leq 1 \\
\Rightarrow R &\leq \frac{(\lambda-1)(\lambda-2)(\lambda-3)}{x^{4-\lambda}} \int_x^\infty e^{-t}\, dt \\
&= \frac{(\lambda-1)(\lambda-2)(\lambda-3)}{x^{4-\lambda}} e^{-x} \\
&\text{which, when divided by } x^{-1}e^{-x} \text{ is O} \left(\frac{1}{\text{x}^4}\right) \\
\text{Therefore I} &= x^\lambda e^{-x} \left(\frac{1}{X} + \frac{(\lambda-1)}{x}\right. \\
&\left. + \frac{(\lambda-1)(\lambda-2)(\lambda-3)}{x^3} + O\left(\frac{1}{x^4}\right)\right)
\end{aligned}
$$

Implied constant $(\lambda-1)(\lambda-2)(\lambda-3)$.