Exam Question
Topic: DiffInt
The function f satisfies the equation

$$f(x) = \frac{\pi}{3} + \int_0^x \cos^2(f(t)\,dt.$$

By differentiating the integral, find a differential equation for $f(x)$. Hence find the function $f(x)$.

Solution

$$f(x) = \frac{\pi}{3} + \int_0^x \cos^2(f(t)\,dt.$$

$$\frac{df}{dx} = \cos^2(f(x)), \quad \text{so} \quad \sec^2(f)\frac{df}{dx} = 1.$$

Therefore $\tan(f(x)) = x + c$.
When $x = 0,\ f(x) = \frac{\pi}{3}$, so $c = \sqrt{3}$.
Thus $f(x) = \tan^{-1}\left(x + \sqrt{3}\right)$.