Question Unlike sequences, the convergence of series whose terms are products and quotients of convergent series does not necessarily follow. Exploring this phenomenon is the purpose of this example. Construct examples of each of the following, or prove that no such example exists:

1. convergent series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ with positive terms for which the series of products $\sum_{n=0}^{\infty} a_n\, b_n$ converges;

2. divergent series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ with positive terms for which the series of products $\sum_{n=0}^{\infty} a_n\, b_n$ diverges;

3. divergent series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ with positive terms for which the series of products $\sum_{n=0}^{\infty} a_n\, b_n$ converges;

4. convergent series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ with positive terms for which the series of quotients $\sum_{n=0}^{\infty} \frac{a_n}{b_n}$ diverges;

5. convergent series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ with positive terms for which the series of quotients $\sum_{n=0}^{\infty} \frac{a_n}{b_n}$ converges;

6. divergent series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ with positive terms for which the series of quotients $\sum_{n=0}^{\infty} \frac{a_n}{b_n}$ diverges;

7. divergent series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ with positive terms for which the series of quotients $\sum_{n=0}^{\infty} \frac{a_n}{b_n}$ converges;

Answer

1. All we need are two convergent series. For instance, take $a_n = (0.5)^n$ and $b_n = (0.3)^n$ for all $n \geq 0$. Then, $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, and $\sum_{n=0}^{\infty} a_n\, b_n$ are all convergent geometric series.

2. take $a_n = 1$ for all $n \geq 0$ and $b_n = 1$ for all $n \geq 0$. Then, both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both divergent geometric series, as is $\sum_{n=0}^{\infty} a_n\, b_n$ (since $a_n\, b_n = 1$ for all $n \geq 0$).

3. for this one, let's take $a_n = b_n = \frac{1}{n}$ for all $n \geq 1$. Then, both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are the harmonic series, and hence divergent. However, the series of products $\sum_{n=1}^{\infty} a_n\, b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, by the note below.

4. take any convergent series, for example $\sum_{n=0}^{\infty} (0.5)^n$, and set $a_n = b_n = (0,5)^n$. Then, the series of quotients is $\sum_{n=0}^{\infty} \frac{a_n}{b_n} = \sum_{n=0}^{\infty} 1$, which diverges.

5. here, we can take $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n^4}$ for $n \geq 1$. Then, both $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^4}$ converge by the note below, as does the series of quotients, as $\frac{a_n}{b_n} = \frac{1}{n^2}$.

6. let's use geometric series again: both of $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} 6^n$ and $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} 2^n$ are divergent geometric series, and the series of quotients $\sum_{n=0}^{\infty} \frac{a_n}{b_n} = \sum_{n=0}^{\infty} 3^n$ is also a divergent geometric series.

7. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 1$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} n^2$ both diverge, but the corresponding sequence of quotients $\sum_{n=1}^{\infty} \frac{a_n}{b_n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Note
The series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges if and only if $s > 1$.

For $s = 1$, this series is called the **harmonic series**, and we can prove directly that it diverges. Note that $\frac{1}{3} + \frac{1}{4} > \frac{1}{2}$, that $\frac{1}{5} + \cdots + \frac{1}{8} > 4\frac{1}{8} = \frac{1}{2}$, and in general that

$$\frac{1}{2^{k-1}+1} + \frac{1}{2^{k-1}+2} + \cdots + \frac{1}{2^k} > 2^{k-1}\frac{1}{2^k} = \frac{1}{2}.$$

Hence, the $(2^k)^{th}$ partial sum S_{2^k} satisfies $S_{2^k} > 1 + k\frac{1}{2}$. Since the terms in the harmonic series are all positive, the sequence of partial sums is monotonically increasing, and by the calculation done the sequence of partial sums is unbounded, and so the sequence of partial sums diverges. Hence, the harmonic series diverges.