Question

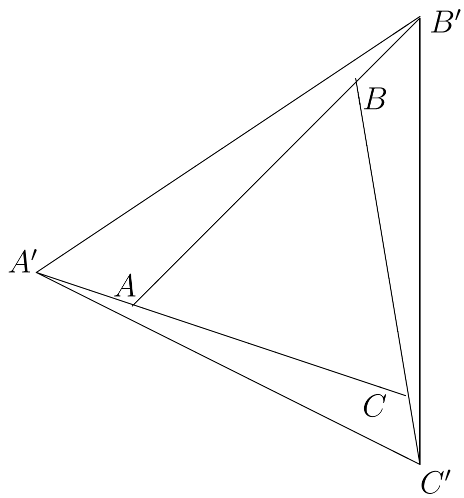
(a) Find the cartesian equation of the plane containing the point $(1, 0, -1)$ parallel to both the lines

$$\frac{x-1}{2} = y = \frac{z+2}{3}$$

$$\frac{x+4}{3} = \frac{y-1}{2} = z$$

Find the shortest distance between each line and the plane.

(b) The triangle ABC has each side extended by the same factor λ, as in the diagram. (So $|AB'| = \lambda|AB|$ etc.) Find the ratio of the areas of triangles $A'B'C'$ and ABC.



Answer

(a) A normal to the plane is

$$(2, 1, 3) \times (3, 2, 1) = (-5, 7, 1)$$

So the plane has equation

$$-5x + 7y + z = k$$

It contains $(1, 0, -1)$ so $k = -6$

Therefore the equation is

$$-5x + 7y + z = -6$$

Distance of $\mathbf{p}$ from $\mathbf{a}\cdot\mathbf{n} = k$ is $\dfrac{|\mathbf{a}\cdot\mathbf{p} - k|}{|\mathbf{a}|}$

So the distance of l_1 from π is the distance of $(1, 0, -2)$ from π

$= \dfrac{|-7+6|}{\sqrt{75}} = \dfrac{1}{\sqrt{75}} = \dfrac{1}{5\sqrt{3}}$

So the distance of l_2 from π is the distance of $(-4, 1, 0)$ from π

$= \dfrac{|27+6|}{\sqrt{75}} = \dfrac{33}{\sqrt{75}} = \dfrac{11\sqrt{3}}{5}$

(b) Let A be the origin $\vec{AB} = \mathbf{b}$ $\vec{AC} = \mathbf{c}$

$$\begin{aligned}
\vec{A'B'} = \vec{A'A} + \vec{AB'} &= (\lambda - 1)\mathbf{c} + \lambda\mathbf{b} \\
\vec{A'C'} = \vec{A'C} + \vec{CC'} &= \lambda c + (\lambda - 1)(\mathbf{c} - \mathbf{b}) \\
&= (2\lambda - 1)\mathbf{c} - (\lambda - 1)\mathbf{b}
\end{aligned}$$

The area of $A'B'C'$ is

$$\begin{aligned}
\frac{1}{2}|(\vec{A'B'} \times \vec{A'C'})| &= \frac{1}{2}\left|((\lambda - 1)\mathbf{c} + \lambda\mathbf{b}) \times ((2\lambda - 1)\mathbf{c}(\lambda - 1)\mathbf{b})\right| \\
&= \frac{1}{2}| + (\lambda - 1)^2 + \lambda(2\lambda - 1)||\mathbf{B} \times \mathbf{C}| \\
&= \frac{1}{2}|3\lambda^2 - 3\lambda + 1|\,|\mathbf{b} \times \mathbf{c}|
\end{aligned}$$

Therefore the required ration is $|3\lambda^2 - 3\lambda + 1|$