Question

(a) Left $x = e^{i\theta}$ $(\theta \neq 0)$. Prove that

$$\frac{z+1}{z-1} = -i\cot\frac{1}{2}\theta.$$

Find all the values of z satisfying $z^7 = -1$, and hence solve the equation $(w+1)^7 = -(w-1)^7$. Deduce the roots of $w^6 + 21x^4 + 35w^2 + 7 = 0$

(b) Sketch the region of the complex plane satisfying the two conditions $1 < |z| < 2$, $0 \leq \arg z \leq \frac{\pi}{4}$

Answer

(a) $\dfrac{e^{i\theta}+1}{e^{i\theta}-1} = \dfrac{e^{\frac{1}{2}i\theta} + e^{-\frac{1}{2}i\theta}}{e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta}} = \dfrac{\cos\frac{1}{2}\theta}{i\sin\frac{1}{2}\theta} = -i\cot\dfrac{1}{2}\theta$

Solutions of $z^7 = -1$ are

$$z = \exp\left(\frac{2k+1}{7}\pi i\right) \qquad k = 0, 1, \cdots, 6$$

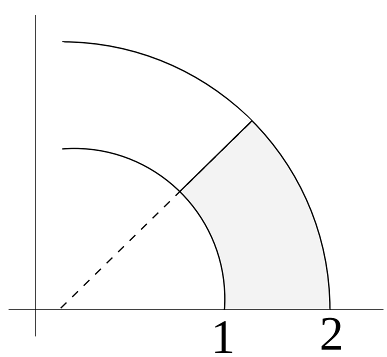
So $\dfrac{w+1}{w-1} = z$

$$w = -i\cot\left(\frac{k+\frac{1}{2}}{7}\pi i\right) \qquad k = 0, 1, \cdots, 6$$

$(w+1)^7 + (w-1)^7 = 0$ reduces to

$2w(w^6 + 21w^4 + 35w^2 + 7) = 0$

So w $= 0$ or $x = -i\cot\dfrac{k+\frac{1}{2}}{7}\pi i \quad k = 0, 1, 2, 4, 5, 6$



(b)