QUESTION The number of cars passing along a road in 3 different five minute periods are recorded as n_1, n_2 and n_5. They may be assumed to be independent observations from Poisson distributions with means $\mu, \lambda\mu$ and $\lambda^2\mu$ respectively. Show that the maximum likelihood estimates $\hat{\lambda}$ and $\hat{\mu}$ satisfy:

$$\begin{aligned} \hat{\mu} + \hat{\lambda}\hat{\mu} + (\hat{\lambda})^2\hat{\mu} &= n_1 + n - 2 + n_3 \\ \hat{\lambda}\hat{\mu} + 2(\hat{\lambda})^2\hat{\mu} &= n_2 + 2n_3 \end{aligned}$$

Find $\hat{\lambda}$ and $\hat{\mu}$ when $n_1 = 30,\ n_2 = 40$ and $n_3 = 50$.

ANSWER $L(\lambda, \mu) = \frac{e^{-\mu}\mu^{n_1}}{n_1!}\frac{e^{-\lambda\mu}(\lambda\mu)^{n_2}}{n_2!}\frac{e^{-\lambda^2\mu}(\lambda^2\mu)^{n_3}}{n_3!}$

$\ln L(\lambda, \mu) = k - (\mu + \lambda\mu + \lambda^2\mu) + (n_1 + n_2 + n_3)\ln\mu + (n_2 + 2n_3)\ln\lambda$

$\frac{\partial \ln L(\lambda,\mu)}{\partial\lambda} = -\mu - 2\lambda\mu + \frac{n_2+2n_3}{\lambda}$

$\frac{\partial \ln L(\lambda,\mu)}{\partial\mu} = -1 - \lambda - \lambda^2 + \frac{n_1+n_2+n_3}{\mu}$

hence mle's satisfy $\hat{\lambda}\hat{\mu} + 2\hat{\lambda}^2\hat{\mu} = n_2 + 2n_3$

$\hat{\mu} + \hat{\lambda}\hat{\mu} + \hat{\lambda}^2\hat{\mu} = n_1 + n_2 + n_3$