FUNCTIONAL ANALYSIS
THE STONE-WEIERSTRASS THEOREM

Let X be a compact space and let A be an algebra of real-valued continuous functions on X which separates the points of X (i.e. $x \neq y \Rightarrow \exists f \in A : f(x) \neq f(y)$) and which has the property that there is no point of X at which all the functions of A vanish. Then A is uniformly dense on the set of all the real-values continuous functions defined on X.

Lemma 1 The set of all continuous functions forms a lattice. Let A be a set of real-valued continuous functions on a compact space X which is closed under the lattice operations $f \vee g$ and $f \wedge g$. Then the uniform closure of A contains every continuous function on X which can be approximated at every pair of points by a function of A.

[The uniform closure of A means the set of functions to which functions of A converge uniformly.]

Proof Let f be any function which can be so approximated and let $\varepsilon > 0$.

Given $x, y \in X$ let $f_{xy} \in A$ be such that $|f_{xy}(x) - f(x)| < \varepsilon$ and $|f_{xy}(y) - f(y)| < \varepsilon$

Fixing y, let $U_{xy} = \{z : f_{xy}(z) < f(z) + \varepsilon\}$

Let $V_{xy} = \{z : f_{xy}(z) > f(z) - \varepsilon\}$

Now $x \in U_{xy}$ therefore $\cup_x U_{xy} \supset X$.

hence there are a finite number of these sets say $U_{x_1 y} \dots U_{x_n y}$ whose union covers X.

We put $f_y = f_{x_1} y \wedge f_{x_2} y \wedge \dots \wedge f_{x_n} y$

$f_y \in A$ as A is closed under $\wedge$.

Write $Vy = Vx_1 y \cap Vx_2 y \cap \dots \cap V_{x_n} y$.

Vy is then a neighbourhood of y and $fy < f + \varepsilon$ everywhere on X $fy > f - \varepsilon$ on the neighbourhood Vy of y as $y \in Vy$. $\cup_y V_y \supset X$, so there is a finite number of these sets, say $V_{y_1} \dots, V_{y_k}$ whose union covers X.

We put $g = fy_1 \vee fy_2 \vee \dots \vee fy_k$.

Then $g \in A$ and $f - \varepsilon < g < f + \varepsilon$ everywhere on X.

Lemma 2 A uniformly closed algebra A of bounded real-valued functions on a set is also closed for the lattice operations.

Proof

$$\begin{aligned} f \vee g &= \frac{f+g+|f-g|}{2} \\ f \vee g &= \frac{f+g-|f-g|}{2} \end{aligned}$$

Hence it suffices to show that $f \in A \Rightarrow |f| \in A$.

We may suppose without loss of generality

$$\|f\| = \sup\{|f(x)| : x \in X\} \leq 1$$

The Taylor series for $(t+\varepsilon^2)^{\frac{1}{2}}$ about $t = \frac{1}{2}$ converges uniformly in $0 \leq t \leq 1$ therefore putting $t = x^2$ there is a polynomial $P(x^2)$ in x^2 such that

$$|P(x^2) - (x^2+\varepsilon^2)^{\frac{1}{2}}| < \varepsilon \text{ on } [-1\ 1]$$

If $Q = P - P(0)$ then since $|P(0)| \leq 2\varepsilon$ we have

$$|Q(x^2) - (x^2-\varepsilon^2)^{\frac{1}{2}}| < 3\varepsilon \text{ on } [-1\ 1]$$

Now $0 < (x^2+\varepsilon^2)^{\frac{1}{2}} - |x| < \varepsilon$ so

$$|Q(x^2) - |x|| < 4\varepsilon \text{ on } [-1\ 1]$$

Since $Q(f^2) \in A$ and $|Q(f^2) - |f|| < 4\varepsilon$ everywhere on X therefore $|F| \in A$ as A is uniformly closed.

Proof of Theorem Let $\overline{A}$=uniform closure of A then it is clear that $\overline{A}$ is an algebra therefore by Lemma 2 it is closed under the lattice operations.

Using the given properties of A we can find a function $g \in A$ so that

$$g(x) \neq 0 \quad g(y) \neq 0 \quad g(x) \neq g(y).$$

then $g(x)g^2(y) \neq g(y)g^2(x)$. Thus we can always find α β satisfying

$$\begin{aligned} \alpha f(x) + \beta f^2(x) &= a \\ \alpha g(y) + \beta g^2(y) &= b \end{aligned}$$

for any given a and b. Hence any f can be approximated at a pair of points by a function of $\overline{A}$ (as $\alpha g + \beta g^2 \in \overline{A}$). Hence the result follows from lemma 1.

Linear Transformations Let E and F be two vector spaces. A transformation T from E to F is linear if $A(alphax + \beta y) = \alpha T(x) + \beta T(y)$ for any $x, y \in E$ and any $\alpha,\ \beta$.

The set of all linear transformations from E to F is itself a vector space over the same field of scalars, for if T_1, T_2 are two such transformations we can define T_1 and T_2 and αT_1 by

$$\begin{aligned} (T_1 + T_2)x &= T_1 x + T_2 x \\ (\alpha T_1)x &= \alpha(T_1 x) \end{aligned}$$

The linear transformations from a vector space onto itself form an algebra, for if $T_1,\ T_2$ are 2 such transformations we can define $T_1 T_2 x = T_1(T_2 x)$.

Continuous linear transformations between Banach Spaces Let E and F be Banach Spaces and let T be a linear transformation from E to F. The following statements are equivalent:

(i) T is continuous.

(ii) T is continuous at one point.

(iii) T is bounded on the unit sphere.

(iv) There is a number N such that $\|Tx\| \leq N\|x\|$ for any $x \in E$.

The set of all continuous linear transformations form a Banach Sphere.

Define $\|T\| = \sup_{\|X\|=1} \|Tx\| = \sup_x \frac{\|Tx\|}{\|x\|}$.

This is clearly a norm.

Suppose $\{T_n\}$ is a Cauchy sequence. For any $x \in E$ $\|T_n(x) - T_m(x)\| \leq \|T_n - T_m\|\|x\|$ therefore $\{T_n(x)\}$ for every x is a Cauchy sequence therefore $T_n(x) \to T(x)$.

Now suppose without loss of generality $T_n(x) \to 0$ for every x. R.T.P. $\|T_n\| \to 0$.

Given $\varepsilon > 0$ choose N such that $\|T_n - T_m\| < \frac{\varepsilon}{2}$ whenever $m, n > N$.

Let $\|x\| \leq 1$. Let $m > N$. Since $T_n(x) \to 0 \exists n > N$ such that $\|T_n(x)\| < \frac{\varepsilon}{2}$

$$\begin{aligned} \|T_m(x) - T_n(x)\| \leq \|T_m - T_n\|\|x\| &\leq \frac{\varepsilon}{2} \\ \text{therefore } \|T_m(x)\| \leq \|T_n(x) - T_m(x)\| + \|T_n(x)\| &< \varepsilon \end{aligned}$$

if we have bounded linear mappings from $X \to$ Complex numbers, the Banach space of these mappings is called the dual space X^* of X. Its elements are called functionals. We may also regard the elements of X as functions defined on X^*.

If $x \in X$ and $x^* \in X^*$ we use $\langle x, x^* \rangle$ for $x^*(x)$.

If E is a vector space, E is a linear subspace of deficiency 1 if $\exists x \in E$ such that $H + [x] = E$.

Suppose f is any functional $f : X \to \mathcal{C}$.

$H = \{x \in E : f(x) - 0\}$ is a hyperplane if $f \not\equiv 0$.

Chooses x_0 such that $f(x_0) \neq 0$.

Let $y \in E$. Then $y - \frac{f(y)}{f(x_0)} x_0 \in H, \quad y = h + \lambda x_0$.

Conversely given any hyperplane $H \exists x$ such that $H + [x] = E$ i.e. $y = h + \lambda x$ λ is unique.

Define $f(y) = \lambda \alpha \quad \alpha \neq 0$ fixed.

Two functionals have the same null space $\Leftrightarrow$ one is a multiple of the other. The continuous functionals are those for which the null space is a closed hyperplane.

If X is a finite dimensional space X^* is the same as X.

Example

$$(\ell^P)^* = \ell^p$$

Let $\{b_n\} \in \ell^q$ then we can define

$$f(\{a_n\}) = \sum a_n b_n \leq \left(\sum |a_n|^P \right)^{\frac{1}{p}} (|sum|b_n|^q)^{\frac{1}{q}} < \infty.$$

$|f(\{a_n\}) \leq \|a_n\|_p \|b_n\|_q$ therefore $\|f\| \leq \|b_n\|_q$.

To show $\|f\| = \|\{b_n\}\|_{\_}q$:

Choose any N and define

$$A_n = |B_n|^{q-1}\frac{\overline{b}_n}{|b_n|}\ n \le N\ 0, n > N$$

Then

$$\begin{aligned} |f(\{a_n\})| &= \sum_1^N |b_n|^q \le \|f\|\|a_n\|_p \\ &= \|f\|\left(\sum_1^N |b_n|^{qp-p}\right)^{\frac{1}{p}} = \|f\|\left(\sum_1^N |b_n|^q\right)^{\frac{1}{p}} \\ \text{therefore } \left(\sum_1^N |b_n|^q\right)^{\frac{1}{q}} &\le \|f\| \end{aligned}$$

therefore $\left(\sum_1^\infty |b_n|^q\right)^{\frac{1}{q}} \le \|f\|$

Now let $f \in (\ell^p)^*$.

Define $b_n = f(\{0, 0, \ldots, 0, , 1, 0, 0 \ldots\})$ where the 1 is in the nth place.

By linearity

$$|f(\{a_1 a_2 \ldots a_n 0, 0, \ldots\})| = \left|\sum_1^N a_n b_n\right| \le \|f\|\left(\sum_1^N |a_n|^P\right)^{\frac{1}{p}}$$

Now choose $a_n = b_n^{q-1}\frac{\overline{b_n}}{|B_n|}$. Then

$$\left|\sum_1^N a_n b_n\right| = \sum |b_n|^q \le \|f\|\left(\sum_1^N |b_n|^{qp-p}\right)^{\frac{1}{p}}$$

Therefore

$$\left(\sum_1^N |b_n|^q\right)^{\frac{1}{q}} \le \|f\|$$

therefore letting $N \to \infty$ it follows that $\{b_n \in \ell^q$ and is that sequence from which f arises.

$(\ell^P)^{**} = (\ell^q)^* = \ell^p$ - reflexive.

$(\ell^1)^* = \ell^\infty$.