Question

a) Let $f(z) = \sum_{n=-k}^{\infty} a_n(z-b)^n$ be the Laurent series for a function analytic in the region $0 < |z-b| < R$ and k is a non-negative integer or ∞. In terms of this series explain what is meant by

i) a removable singularity,

ii) an isolated essential singularity,

iii) a pole of order m.

Classify all the singularities of the function

$$\frac{(z^2-1)e^{\frac{1}{z}}}{(z^2-3z+2)\sin(z+1)}.$$

b) Find the Laurent expansions of

$$\frac{1}{z^2(1-z)}$$

valid for

i) $0 < |z| < 1$,

ii) $|z| > 1$,

iii) $0 < |z-1| < 1$.

Answer

a) Bookwork

$f(z) = \dfrac{(z+1)(z-1)e^{\frac{1}{z}}}{(z-1)(z-2)\sin(z+1)}$

Removable singularity at $z = 1$

Removable singularity at $z = -1$ because $\dfrac{z+1}{\sin(z+1)} \to 1$ as $z \to -1$

Simple pole at $z = 2$

Essential singularity at $z = 0$ due to $e^{\frac{1}{z}}$.

b) i) In $0 < |z| < 1$

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots$$

so $$\frac{1}{z^2(1-z)} = \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \cdots$$

ii) for $|z| > 1$

$$\frac{1}{z^2}\frac{1}{1-z} = \frac{1}{z^2}\frac{-1}{z\left(1-\frac{1}{z}\right)} = \frac{-1}{z^3}\frac{1}{1-\frac{1}{z}}$$

$$= \frac{-1}{z^3}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right) = -\frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^6} - \cdots$$

iii) for $0 < |z-1| < 1$

$$\frac{1}{z^2} = \frac{1}{[1-(1-z)]^2} = 1 + 2(1-z) + 3(1-z)^2 + 4(1-z)^3 + \cdots$$

so $$\frac{1}{z^2(1-z)} = \frac{-1}{z-1} + 2 - 3(z-1) + 4(z-1)^2 - \cdots$$