QUESTION

The arithmetic function i is defined by $i(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$. Prove

(i) i is multiplicative.

(ii) $i * f = f * i = f$ for all arithmetic function f.

(iii) If $f(1) \neq 0$, we can find an arithmetic function g satisfying $f * g = g * f = i$.

[Note: from questions 6 and 7, we can see that the set of all arithmetic functions f with $f(1) \neq 0$ form an abelian group under *. This allows us to apply results from group theory to the study of arithmetic functions.]

ANSWER

(i) If $\gcd(m, n) = 1$, then either $m = n = 1$, or at least one of m and n is > 1. In the latter case $mn > 1$. Thus $i(mn) = \begin{cases} 1 & \text{if } m = n = 1 \\ 0 & \text{otherwise} \end{cases}$

But if at least one of m, n is > 1, then at least one of $i(m)$, $i(n)$ is equal to 0, and so their product is 1. Thus in all cases $i(mn) = i(m)i(n)$ and i is multiplicative.

(ii) By part (ii) of question 6, $i * f = f * i$, so it will be enough to proof $i * f = f$.

Now $i * f(n) = \sum_{d|n} i(d) f\left(\frac{n}{d}\right)$. But $i(d) = 0$ unless $d = 1$, so the only term contributing to this sum is the first, so we get $i * f(n) = i(1) f\left(\frac{n}{1}\right) = 1.f(n) = f(n)$, This is true for all values of n, so $i * f = f$, as required.

(iii) Again, question 6(ii) tells us that we need only find g such that $f * g = i$. As g is an arithmetic function, to describe g we need to specify its values on the natural numbers. We will find g by describing $g(1), g(2), g(3), \ldots$ and eventually getting $g(n)$ in terms of the values already spacified for $g(k)$ with $k < n$.

We first want $f * g(1) = i(1) = 1$. Now $(f * g)(1) = \sum_{d|1} f(d) g\left(\frac{1}{d}\right)$ and as the onlt divisor of 1 is 1, this says $(f * g)(1) = f(1)g(1)$. Thus if we define $g(1) = f(1)^{-1}$ (allowable as $f(1) \neg 0$), we will have $f * g(1) = i(1)$. Next we want $f * g(2) = i(2) = 0$, We have $f * g(2) = \sum_{d|2} f(d) g\left(\frac{2}{d}\right) = f(1)g(2) + f(2)g(1)$. We have already defined $g(1)$, so we may define $g(2) = \frac{-f(2)g(1)}{f(1)} = \frac{-f(2)}{f(1)^2}$. This ensures that $f * g$ and i agree at 1 and 2.

Similarly we want $f * g(3) = i(3) = 0$, i.e. $f(1)g(3) + f(3)g(1) = 0$, so again define $g(3) = \frac{-f(3)g(1)}{f(1)} = \frac{-f(3)}{f(1)^2}$.To get $f * g(4) = i(4) = 0$ we need $0 = f * g(4) = \sum_{d_4} f(d)g\left(\frac{4}{d}\right) = f(1)g(4) + f(2)g(2) + f(4)g(1)$. (as the divisors of 4 are 1, 2 and 4). This will give $g(4) = \frac{-f(2)g(2) - f(4)g(1)}{f(1)}$, and as $g(1)$ and $g(2)$ are already defined this tells us how to define $g(4)$.

We continue in this way until $g(1), g(2), \ldots g(n-1)$ have all been defined. To define $g(n)$ we note that we want $f * g(n) = i(n) = 0$, so we want $0 = \sum_{d|n} f(d)g\left(\frac{n}{d}\right) = f(1)g(n) + \sum_{d>1, d|n} f(d)g\left(\frac{n}{d}\right)$. Now all the terms in the sum $\sum_{d>1, d|n} f(d)g\left(\frac{n}{d}\right)$ are already specified, so we may now define $g(n) = \frac{-1}{f(1)} \sum_{d>1, d|n} f(d)g\left(\frac{n}{d}\right)$.

In this way the value of $g(n)$ is defined inductively for all n, giving us an arithmetic function g with the properties required.

[It is worth noting that an arithmetic function is any function whose domain in N, The functions d, σ, σ_1 etc, that we've considered happen to take values in N, but this is not a general requirement. If the function f we start with here has $f(1) \neq 1$, the values of g will not be integers, but this does not matter.]