Question

In a simple random walk on $0, 1, 2, \ldots, a$ there are probabilities p, q and $1-p-q$ of a step of $1, -1$ and 0 respectively . There is a reflecting barrier at 0 defined by:

$$P\{X_n = 0 | X_{n-1} = 0\} = 1 - p,$$
$$P\{X_n = 1 | X_{n-1} = 0\} = p,$$

and an absorbing barrier at a.

Derive a difference equation for the expected number, E_z, of steps from a start at $z \quad (0 \leq z \leq a)$ until absorption occurs. Find E_z for the two cases $p = q$ and $p \neq q$.

What is the difference in the expected duration between walks starting at 0 and starting at 1?

Answer

For $0 < z < a$,

$E_z = p(1 + E_{z+1}) + q(1 + E_{z-1}) + (1 - p - q)(1 + E_z)$

$\Rightarrow pE_{z+1} - (p+q)E_z + qE_{z-1} = -1$

The auxiliary equation is $p\lambda^2 - (p+q)\lambda + q = 0 \Rightarrow \lambda = 1, \ \dfrac{q}{p}$

For $p \neq q$ a particular solution is $E_z = cz$ and substituting in the difference equation gives $c = \dfrac{1}{q-p}$

So the general solution is

$$E_z = A + B\left(\frac{q}{p}\right)^z + \frac{z}{q-p} \qquad 0 \leq z \leq a$$

Boundary conditions:

$E_a = 0$ so $0 = A + B\left(\dfrac{q}{p}\right)^a + \dfrac{a}{q-p}$

$E_0 = p(1 + E_1) + (1-p)(1 + E_0) \Rightarrow 1 + pE_1 - pE_0 = 0$

So $1 + p\left(A + B\left(\dfrac{q}{p}\right)\dfrac{1}{q-p}\right) - p(a + B) = 0$

Solving the two equations for A and B gives

$$E_z = \underbrace{\frac{q}{(q-p)^2}\left(\frac{q}{p}\right)^a - \frac{a}{q-p}}_{A} \underbrace{-\frac{q}{(q-p)^2}}_{B}\left(\frac{q}{p}\right)^z + \frac{z}{q-p}$$

For $p = q$ the auxiliary equation has equal roots $\lambda = 1$, so the general solution is $A + Bz$. A particular solution will be cz^2, and substituting gives $c = -\dfrac{1}{2p}$

So $E_z = A + Bz - \frac{z^2}{2p}$

The boundary conditions are as above, and give

$(z = a) \quad 0 = A + Ba - \frac{a^2}{2p}$

$(z = 0) \quad 1 = p\left(A + B - \frac{1}{2p}\right) - p \cdot A = 0$

So $E_z = \frac{a^2 + a}{2p} - \frac{z}{2p} - \frac{z^2}{2p}$

from which $E_0 - E_1 = \frac{1}{p}$

Alternatively this equation is simply a rearrangement of the boundary condition at $z = 0$