Question

Explain what is meant by a population with size varying according to a generalized birth-death process having birth and death rates given by the functions $\Gamma_1(n)$ and $\Gamma_2(n)$ respectively. A population of this kind has size varying between N and M and

$$\Gamma_1(n) = \lambda(N-n)n \quad \Gamma_2(n) = \mu(n-M)n,$$

where $N > M > 0$ and λ and μ are positive constants. Show that the probability $p_n(t)$ that the population size is n at time t satisfies the following differential-difference equation for $n = M+1, M+2, \ldots, N-1$.

$$\begin{aligned} p_n'(t) &= \lambda(N-n+1)(n-1)p_{n-1}(t) + \mu(n+1-M)(n+1)p_{n+1}(t) \\ &\quad -[\lambda(N-n)n + \mu(n-M)n]p_n(t). \end{aligned}$$

Obtain corresponding equations for $p_N'(t)$ and $p_M'(t)$.
If $X(t)$ denotes the population size at time t show that

$$\frac{d}{dt}E\{X(t)\} = (M\mu + N\lambda)E\{X(t)\} - (\mu+\lambda)E\{[X(t)]^2\}$$

where $E\{X(t)\}$ denotes the expected value of $X(t)$.

Answer

Suppose we have a population of individuals reproducing or dying independently of one another. Suppose the size of the population at time t is $X(t)$. Then a generalized birth - death process with rates $\Gamma_1(n)$ and $\Gamma_2(n)$ is defined by the probabilities:

$$\begin{aligned} P(X(t+\delta t) = n+1 | X(t) = n) &= \Gamma_1(n)\delta t + o(\delta t) \\ P(X(t+\delta t) = n-1 | X(t) = n) &= \Gamma_2(n)\delta t + o(\delta t) \\ P(X(t+\delta t) = n | X(t) = n) &= 1 - (\Gamma_1(n) + \Gamma_2(n))\delta t + o(\delta t) \end{aligned}$$

For $M < n < N$ we have

$$\begin{aligned} p_n(t+\delta t) &= P(X(t+\delta t) = n | X(t) = n+1)P(X(t) = n+1) \\ &+ P(X(t+\delta t) = n | X(t) = n-1)P(X(t) = n-1) \\ &+ P(X(t+\delta t) = n | X(t) = n)P(X(t) = n) \\ &= (\Gamma_2(n+1)\delta t + o(\delta t))p_{n+1}(t) \\ &+ (\Gamma_1(n-1)\delta t + o(\delta t))p_{n-1}(t) \\ &+ (1 - (\Gamma_1(n) + \Gamma_2(n)\delta t + o(\delta t))p_n(t) \end{aligned}$$

Thus

$$\begin{aligned}\frac{p_n(t+\delta t)-p_n(t)}{\delta t} &= \Gamma_2(n+1)p_{n+1}(t)+\Gamma_1(n-1)p_{n-1}(t)\\ &- (\Gamma_1(n)+\Gamma_2(n))p_n(t)+\frac{o(\delta t)}{\delta t}\end{aligned}$$

Thus

$$\begin{aligned}p_n'(t) &= \Gamma_2(n+1)p_{n+1}(t)+\Gamma_1(n-1)p_{n-1}(t)-(\Gamma_1(n)-\Gamma_2(n))p_n(t)\\ &= \mu(n+1-M)(n+1)p_{n+1}(t)+\lambda(N-n+1)(n-1)p_{n-1}(t)\\ &- [\lambda(N-n)n+\mu(n-M)n]p_n(t) \qquad (1)\end{aligned}$$

By reasoning similar to that above we also obtain:

$$\begin{aligned}p_N'(t) &= \lambda(N-1)p_{N-1}(t)-\mu(N-M)Np_N(t) \qquad (2)\\ p_M'(t) &= \mu(M+1)p_{M+1}(t)-\lambda(N-M)Mp_M(t) \qquad (3)\end{aligned}$$

Now $\frac{d}{dt}E(X(t))=\sum_{n=M}^{N} np_n'(t)$, so summing (1), (2) and (3) gives:

$$\sum_{n=M}^{N-1}\mu(n+1-M)n(n+1)p_{n+1}(t)-\sum_{n=M}^{N}n(\lambda(N-n)n+\mu(n-M)n)p_n(t)$$

$$+\sum_{n=M+1}^{N}\lambda(N-n+1)n(n-1)p_{n-1}(t)$$

Changing the index of summation in the first and last sums gives

$$\sum_{n=M+1}^{N}\mu(n-M)(n-1)np_n(t) \qquad \leftarrow \text{summand} = 0 \text{ for } n=M$$

$$+\sum_{n=M}^{N-1}\lambda(N-n)n(n+1)p_n(t) \qquad \leftarrow \text{ditto for } n=N$$

$$-\sum_{n=M}^{N}n(\lambda(N-n)n+\mu(n-M)n)p_n(t)$$

$$\sum_{n=M}^{N}p_n(t)\times K, \quad \text{where}$$

$$\begin{aligned}K &= \mu(n-M)(n^2-n)-\lambda n^2(N-n)-\mu n^2(n-M)\\ &\quad +\lambda(N-n)(n^2+n)\\ &= (\mu M+\lambda N)n-(\mu+\lambda)n^2\end{aligned}$$

Thus $\frac{d}{dt}E\{X(t)\}=(M\mu+N\lambda)E\{X(t)\}-(\mu+\lambda)E\{[X(t)]^2\}$