Question

(a) A six state Markov chain has the probability transition matrix P given below. Classify the states as positive recurrent, null recurrent or transient. Obtain the mean recurrence times for all positive recurrent states.

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

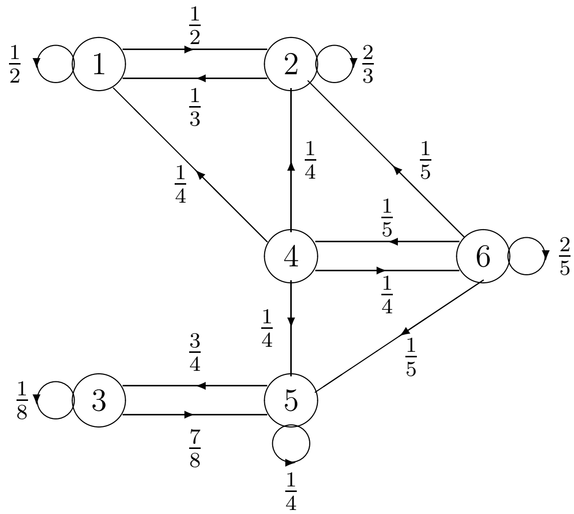
(b) Balls are thrown independently into N boxes so that each ball has a probability $\frac{1}{N}$ of falling into each box. Let X_n be the number of empty boxes after n balls have been thrown. Does $\{X_n\}$ $n = 1, 2, \ldots$ form a Markov chain? If $p_n(\mathrm{k})$ denotes the probability that $X_n = k$ show that

$$p_{n+1}(k) = \left(1 - \frac{k}{N}\right) p_n(k) + \frac{k+1}{N} p_n(k+1) \quad \text{for} \quad k = 0, 1, \ldots, N-1$$

Answer

(a) The transition matrix P is:

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$



$\{1,2\}$ and $\{3,5\}$ are closed sets and constitute subchains which are ergodic.

The 2×2 chain with $P = \begin{pmatrix} 1-p & p \\ \alpha & 1-\alpha \end{pmatrix}$

has equilibrium distribution $\left(\dfrac{\alpha}{\alpha+p}, \dfrac{p}{\alpha+p}\right)$.

Mean recurrence times are $\dfrac{\alpha+p}{\alpha}$ and $\dfrac{\alpha+p}{p}$ so in these cases we have

$$\mu_1 = \frac{5}{2} \quad \mu_2 = \frac{5}{3} \quad \mu_3 = \frac{13}{6} \quad \mu_5 = \frac{13}{7}$$

States 4 and 6 intercommunicate and are of the same type.

$$\begin{aligned} f_{44} &= \frac{1}{4}\cdot\frac{1}{5} + \frac{1}{4}\cdot\frac{2}{5}\cdot\frac{1}{5} + \frac{1}{4}\cdot\left(\frac{2}{5}\right)^2\cdot\frac{1}{5} + \ldots \\ &= \frac{1}{4}\cdot\frac{1}{5}\left(1 + \frac{2}{5} + \left(\frac{2}{5}\right)^2 + \ldots\right) \\ &= \frac{1}{12} < 1 \end{aligned}$$

so 4 and 6 are transient.

(b) $P(X_n = a | X_n = a_{n-1}, \ldots.) = P(X_n = a | X_{n-1} = a_{n-1})$

The number of empty boxes after n balls have been thrown depends only on how many boxes were empty after $(n-1)$ balls had been thrown, together with the result of the n-th throw.

For $k < N$ we have

$$\begin{aligned} P(X_{n+1} = k) &= P(X_n = k \text{ and ball } (n+1) \text{ lands in a} \\ &\qquad\qquad\qquad\qquad \text{non empty box}) \\ &\quad + P(X_n = k+1 \text{ and ball } (n+1) \text{ lands in an} \\ &\qquad\qquad\qquad\qquad \text{empty box}) \\ &= \left(1 - \frac{k}{N}\right) P(X_n = k) + \frac{k+1}{N} P(X_n = k+1) \\ \text{i.e. } p_{n+1} &= \left(1 - \frac{k}{N}\right) p_n(k) + \frac{k+1}{N} p_n(k+1) \end{aligned}$$