Question

Suppose $f : \mathbf{R^n} \to \mathbf{R}^*$ is measurable and $m(S) < \infty$. Suppose also that $A \leq f(x) \leq B$ a.e in S.

Show that f is integrable over S and that $Am(S) \leq \int_S f \leq Bm(S)$

Answer

Let $g(x) = f(x)$ if $A \leq f(x) \leq B$ but $g(x) = A$ otherwise.

Then $\int_S A = \int AX_S \leq \int gX_S = \int_S g \leq \int BX_S = \int_S B$

$Am(S) = \int_S A = \int_S g = \int_S f \leq \int_S B = Bm(S)$