Question

Prove that

$$\cos\theta + \cos 3\theta + \cos 5\theta + ... + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}$$

$$\sin\theta + \sin 3\theta + \sin 5\theta + ... + \sin(2n-1)\theta = \frac{sin^2 n\theta}{\sin\theta}$$

Answer

$$\begin{aligned} e^{i\theta} + e^{3i\theta} + ... + e^{(2n-1)i\theta} &= \frac{e^{i\theta}(1-e^{2ni\theta})}{1-e^{2i\theta}} \\ &= \frac{1-2^{2ni\theta}}{e^{-i\theta}-e^{i\theta}} \\ &= \frac{1-e^{2ni\theta}}{-2i\sin\theta} \\ &= \frac{i(1-\cos 2n\theta - i\sin 2n\theta)}{2\sin\theta} \end{aligned}$$

Similarly

$$e^{i\theta} + e^{3i\theta} + ... + e^{(2n-1)i\theta} = -\frac{i(1-\cos 2n\theta + i\sin 2n\theta)}{2\sin\theta}$$

So

$$\begin{aligned} \cos\theta + ... + \cos(2n-1)\theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta} + ... + e^{(2n-1)i\theta} + e^{-(2n-1)i\theta} \\ &= \frac{1}{2}\left(\frac{i(1-\cos 2n\theta - i\sin 2n\theta)}{2\sin\theta}\right. \\ &\qquad \left. - \frac{(1-\cos 2n\theta + i\sin 2n\theta)}{2\sin\theta}\right) \\ &= \frac{1}{4\sin\theta}(i - i\cos 2n\theta + \sin 2n\theta - \\ &\qquad i + i\cos 2n\theta + \sin 2n\theta) \\ &= \frac{2\sin 2n\theta}{4\sin\theta} \\ &= \frac{\sin 2n\theta}{2\sin\theta} \end{aligned}$$

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\begin{aligned}
\sin\theta + ... + \sin(2n-1)\theta &= \frac{e^{-i\theta}(1-e^{-2n\theta})}{1-e^{-2i\theta}} \\
&= \frac{1}{ai\sin\theta}(i - i\cos 2n\theta + \sin 2n\theta + i \\
&\qquad -i\cos 2n\theta - \sin 2n\theta) \\
&= \frac{i - i\cos 2n\theta}{2i\sin\theta} \\
&= \frac{1-\cos 2n\theta}{2\sin\theta} \\
&= \frac{sin^2 n\theta}{\sin\theta}
\end{aligned}
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