QUESTION

Solve the following system of linear differential equations subject to the initial conditions $\frac{dx}{dt} = 4,\ \frac{dy}{dt} = 20,\ \frac{dz}{dt} = -48$ when

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} 4 & 0 & -1 \\ -1 & 3 & 1 \\ -1 & -1 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

ANSWER

The general solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = k_1 \mathbf{v}_1 e^{\lambda_1 t} + k_2 \mathbf{v}_2 e^{\lambda_2 t} + k_3 \mathbf{v}_3 e^{\lambda_3^t}$$

where $k_1,\ k_2,\ k_3$ are arbitrary real constants, $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$ are the eigenvectors of A and $\lambda_1,\ \lambda_2,\ \lambda_3$ are the corresponding eigenvalues.

When $A = \begin{pmatrix} 4 & 0 & -1 \\ -1 & 3 & 1 \\ -1 & -1 & -5 \end{pmatrix}$ the eigenvalues λ_i satisfy $(4-\lambda)((3-\lambda)(-5-\lambda)$ $+1) - (1 + (3-\lambda))$ or $(4-\lambda)\,((3-\lambda)(-5-\lambda))$ so $\lambda_1 = 4,\ \lambda_2 = 3,\ \lambda_3 = -5$.

Eigenvectors are:

$\lambda_1 = 4:$

$$\begin{pmatrix} 0 & 0 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & -9 \end{pmatrix} \mathbf{v}_1 = \mathbf{0} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$\lambda_2 = 3:$

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ -1 & -1 & -8 \end{pmatrix} \mathbf{v}_2 = \mathbf{0} \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ -9 \\ 1 \end{pmatrix}$$

$\lambda_3 = -5:$

$$\begin{pmatrix} 9 & 0 & -1 \\ -1 & 8 & 1 \\ -1 & -1 & 0 \end{pmatrix} \mathbf{v}_3 = \mathbf{0} \Rightarrow \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 9 \end{pmatrix}$$

so the general solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{4t} + k_2 \begin{pmatrix} 1 \\ -9 \\ 1 \end{pmatrix} e^{3t} + k_3 \begin{pmatrix} 1 \\ -1 \\ 9 \end{pmatrix} e^{-5t}$$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = 4k_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{4t} + 3k_2 \begin{pmatrix} 1 \\ -9 \\ 1 \end{pmatrix} e^{3t} - 5k_3 \begin{pmatrix} 1 \\ -1 \\ 9 \end{pmatrix} e^{-5t}$$

so initial conditions give

$$\begin{pmatrix} 4 \\ 20 \\ -48 \end{pmatrix} = \begin{pmatrix} 4 & 3 & -5 \\ -4 & -27 & 5 \\ 0 & 3 & -45 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

adding rows 1 and 2 gives

$$\begin{pmatrix} 4 \\ 24 \\ -48 \end{pmatrix} = \begin{pmatrix} 4 & 3 & -5 \\ 0 & -23 & 0 \\ 0 & 3 & -45 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \Rightarrow k_2 = -1$$

so $-3 - 45k_3 = -48 \Rightarrow k_3 = 1 \Rightarrow 4k_1 - 3 - 5 = 4 \Rightarrow k_1 = 3$ to give the solution

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} e^{4t} - \begin{pmatrix} 1 \\ -9 \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} 1 \\ -1 \\ 9 \end{pmatrix} e^{-5t}$$