Question
State Rouche's theorem and use it to show that all the roots of the equation

$$z^6 + \alpha z + 1 = 0,$$

where the constant α satisfies $|\alpha| = 1$, lie in the annulus $\frac{1}{2} < |z| < 2$.
Use the argument principle to show that, if $Re(\alpha) > 0$, then just one of these roots lies in the first quadrant.

Answer
If $|z| = \frac{1}{2}$, $\quad |z^6 + \alpha z| \leq |z|^6 + |\alpha||z| < (\frac{1}{1})^6 + \frac{1}{2} < 1$
So $z^6 + \alpha z + 1$ has no roots in $|z| \leq \frac{1}{2}$
If $|z| = 2$, $\quad \alpha z + 1| \leq |\alpha||z| + 1 \leq 3 < 2^6 = |z|^6$
so all the roots are inside $|z| = 2$
DIAGRAM
Let $\alpha = a + ib$, $\quad a > 0$
$f(z) = z^6 + \alpha z + 1$
On OA, $\quad f = x^6 + ax + 1 + ibx$
$\tan\arg f(z) = \dfrac{bz}{x^6 + ax + 1}$ continuous on OA as $a > 0$
This is zero when $x = 0$ and $\to 0$ as $x \to \infty$, so the total change of $\arg f(z)$ on OA is ϵ_1 - small.

On BO $\quad z = iy \quad f = -y^6 + by + 1 + iay$
Consider the real parts $-y^6 - by + 1$, the derivative is $-6y^5 - b$ which is always negative if $b > 0$, and which has just one positive root for $b > 0$. So $-y^6 - by + 1$ has one positive root.
So the graph of $\tan\arg f(z)$ is of the form
DIAGRAM
So as y goes from ∞ to 0, the change of argument of $f(z)$ is $-\pi$.
On the semicircle $Re^{i\theta}$ $\quad f(z) = R^6 e^{i6\theta}(1 + w)$ - w small.
So as $\theta : 0 \to \frac{\pi}{2}$ $\quad \arg f(z)$ increases by approximately 3π.
Thus $[\arg f(z)]_C = 2\pi$ and thus there is just one root in the first quadrant.