## Question

a) Show that the function $f(z) = \bar{z}^2 z$ is differentiable only at $z = 0$.

b) Show that $u = \sin x \sinh y$ is a harmonic function and find a conjugate harmonic function $v$. Express the regular function $u + iv$ in terms of $z = x + iy$.

c) Evaluate $\int_\gamma \log z dz$ where $\gamma$ is the upper half of the unit circle from $z = 1$ to $z = -1$ and the branch of $\log z$ is chosen which is 0 at $z = 1$.

## Answer

a) $f(z) = \bar{z}^2 z = (x - iy)^2(x + iy) = x^3 + xy^2 + i(-x^2y - y^3) = u + iv$

$\frac{\partial u}{\partial x} = 3x^2 + y^2 \qquad \frac{\partial v}{\partial y} = x^2 - 3y^2 \qquad \frac{\partial u}{\partial y} = 2xy \qquad \frac{\partial v}{\partial x} = -2xy$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ iff $3x^2 + y^2 = -x^2 - 3y^2$

i.e. $4x^2 = -4y^2$ i.e $x = y = 0$

So the Cauchy-Riemann equations are satisfied only at $z = 0$. The partial derivatives are continuous there so $f(z)$ is differentiable at $z = 0$.

b) $u = \sin x \sinh y$

$\frac{\partial u}{\partial x} = \cos x \sinh y \qquad \frac{\partial^2 u}{\partial x^2} = -\sin x \sinh y$

$\frac{\partial u}{\partial y} = \sin x \cosh y \qquad \frac{\partial^2 u}{\partial y^2} = \sin x \sinh y$

So $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ i.e $u$ is harmonic.

$\frac{\partial u}{\partial x} = \cos x \sinh y = \frac{\partial v}{\partial y}$

So $v = \cos x \cosh y + \phi(x)$

$-\frac{\partial u}{\partial y} = -\sin x \cosh y = \frac{\partial v}{\partial x}$

So $v = \cos x \cosh y + \psi(y)$

Therefore $v = \cos x \cosh y + k$

Take $k = 0$,

$u + iv = \sin x \sinh y + i \cos x \cosh y$

$= i(\cos x \cos iy - \sin x \sin iy) = i \cos(x + iy) = i \cos z$

c) $\log z = \log |z| + i \arg z$, so for $z = e^{it}$,

$\log z = \log 1 + it = it$

So $\displaystyle\int_C \log z dz = \int_0^\pi itie^{it} dt = -\int_0^\pi te^{it} dt$

$\displaystyle= i \left[\frac{te^{it}}{i}\right]_0^\pi + \frac{1}{i}\int_0^\pi e^{it} dt = -\frac{\pi e^{i\pi}}{i} + \left[-e^{it}\right]_0^\pi$

$= -\pi i + 1 + 1 = 2 - \pi i$