Exam Question
Topic: Chain Rule
The function $z(x, y)$ satisfies the equation

$$x^2\frac{\partial z}{\partial x} - y^2\frac{\partial z}{\partial y} = 0.$$

Transform the equation using the change of variables given by

$$u = \frac{1}{x} + \frac{1}{y}, \quad v = y,$$

and deduce that $z = f\left(\frac{x+y}{xy}\right)$ where f is an arbitrary function.

Solution

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}\left(-\frac{1}{x^2}\right) + \frac{\partial z}{\partial v}.0 = -\frac{1}{x^2}\frac{\partial z}{\partial u}\\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}\left(-\frac{1}{y^2}\right) + \frac{\partial z}{\partial v}.1.\end{aligned}$$

So if

$$x^2\frac{\partial z}{\partial x} - y^2\frac{\partial z}{\partial y} = 0$$

it follows that

$$-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial u} - y^2\frac{\partial z}{\partial v} = 0 \;\text{ so }\; \frac{\partial z}{\partial v} = 0,$$

i.e. z is a function of u only, and so $z = f(u) = f\left(\frac{x+y}{xy}\right)$.