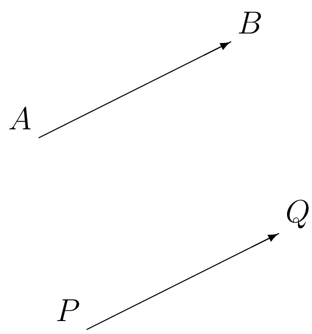
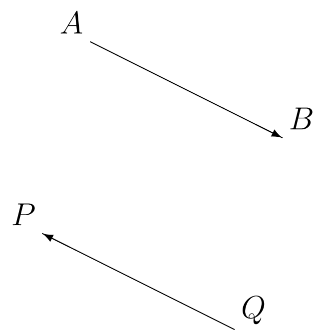
Vector Algebra and Geometry

Scalar and Vector Quantities

A scalar quantity is some physical quantity with which is associated a measure of its magnitude, but no idea of spatial direction, examples being mass, volume, temperature, electrical charge, atmospheric pressure.
A vector quantity is a physical quantity with which is associated a magnitude and a direction in space, examples being velocity, force, momentum.
Many such quantities combine with a similar law of combination, so that the same algebra can be used to describe them. Notice that the physical quantities described above may have other properties than magnitude and direction. For example a force may have a point of application in many situations. However we shall be concerned only with abstracting the properties of magnitude and direction and modelling these.
We shall consider translations or displacements in 2 or 3 dimensions in order to formulate the idea of a vector, for such a translation in space is completely characterised by its magnitude and direction. We can think of a translation as a function which maps each point of space onto another point of space in a particular way. On a diagram or in space such a translation is completely specified by giving the image of any one point.
So in the plane if I speak of the translation which maps $(1, 2)$ onto $(3, 4)$ then we automatically know what happens to every other point in the plane, and similarly with a translation in 3-space.
If a translation takes A to B we can denote it by $\vec{AB}$. Such a translation will map a point P onto a point Q say, so that we might have $\vec{AB} = \vec{PQ}$. This will be the case if the line segments AB and PQ have the same length and direction. We can further characterise the idea of direction by imagining a directed line segment, so that $\vec{AB} = \vec{PQ}$ if the length of AB equals that of PQ, AB is parallel to PQ and the sense of AB is the same as that of PQ.



$\vec{AB} = \vec{PQ}$



$\vec{AB} \neq \vec{PQ}$

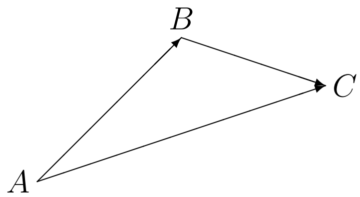
we sometimes say that these are anti-parallel.

Notice that $\vec{AB}$ does not stand for the directed line segment. If it did then clearly $\vec{AB} = \vec{PQ}$ is false as the two line segments are different. Rather $\vec{AB}$ denotes the translation T which takes A to B. Since T also takes P to Q, T can also be symbolised by $\vec{PQ}$.

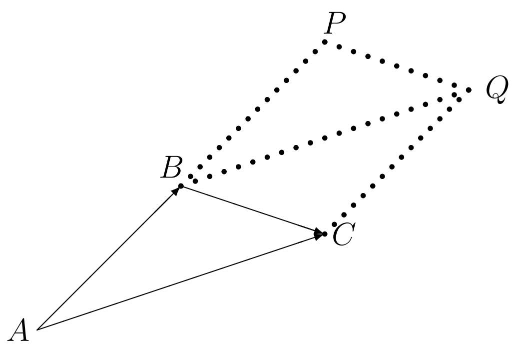
We shall want to use symbols for vectors indendent of any particular line segment used to represent such vectors and so we shall use symbols such as $\mathbf{a}$, $\mathbf{b}$ etc. The magnitude of a vector is the distance of the translation, i.e. the length of any representative line segment. It is called the modulus of the vector, and is denoted by $|\mathbf{a}|$ or just a. In books you will find vectors denoted by heavy type, and their magnitude using the same letter in ordinary type.

Given a vector $\mathbf{a}$ and a vector $\mathbf{b}$ we form their sum as follows. Let A be an arbitrary point in space. Vector $\mathbf{a}$ translates A to some point B, so that $\mathbf{a} = \vec{AB}$. $\mathbf{b}$ then translates B to some point C so that $\mathbf{b} = \vec{BC}$. We then define $\mathbf{a} + \mathbf{b} = \vec{AC}$. This is easily remembered in the form $\vec{AB} + \vec{BC} = \vec{AC}$. We use the symbol $+$ only because the algebraic properties correspond to those of number algebra, as we shall see. It is really translation $\mathbf{a}$ followed by translation $\mathbf{b}$. For if we translate A to B then B to C this has the same effect as translating A straight to C.

In a diagram we have

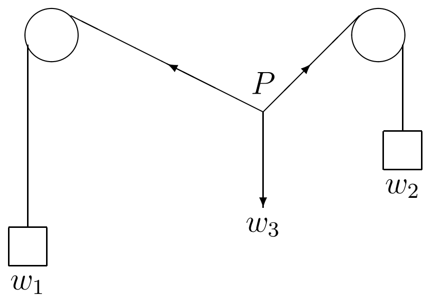


this is therefore sometimes called the triangle law of addition.

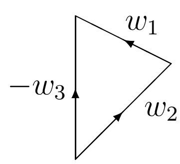


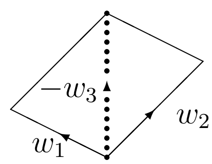
If we draw
$BP = AB$
$PQ = BC \ \ PQ//BC$
$CQ = AB \ \ CQ//AB$
then $BQ = AC$ and $BQ//AC$

so that $\vec{BP} + \vec{BC} = \vec{BQ}$.
It is therefore also called the parallelogram law of addition.
The reason why vector algebra is useful in applied mathematics is that many physical quantities are found experimentally to combine according to this law.
If we let the following system come to equilibrium then we find that the forces at P measured by the weights and direction of string satisfy the triangle of forces. The result of w_1 and w_2 acting on P in the directions shown is equal and opposite to w_3.



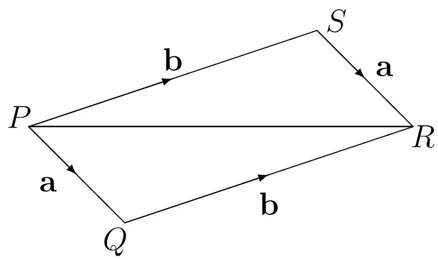
The diagrams below do not reflect the fact that the forces are acting at the same point. They only reflect the magnitude and direction properties.





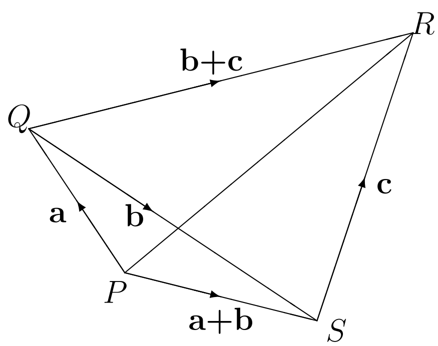
We can see from the diagram that $|\vec{AC}| < |\vec{AB}| + |\vec{BC}|$.
If however A, B, C are in line then $|\vec{AC}| = |\vec{AB}| + |\vec{BC}|$, so generally we have $|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$.
We have a problem if C coincides with A, for then $\vec{AC}$ does not make sense, there is no direction specified. However to make the algebra work we introduce the idea of a zero vector, $\mathbf{0}$ corresponding to the identity transformation, mapping each point to itself. We then have $\mathbf{a}+\mathbf{0} = \mathbf{0}+\mathbf{a} = \mathbf{a}$ for all $\mathbf{a}$.
Also if $\mathbf{a} = \vec{AB}$ then $\vec{AB} + \vec{BA} = \mathbf{0}$, and we write $\vec{BA} = -\mathbf{a}$, by analogy with numbers.
Since vector addition is a new operation with new objects we need to see what the algebra is like

i) commutative law of addition



Consider the parallelogram $PQRS$ where $\vec{PQ} = \mathbf{a}$ $\vec{QR} = \mathbf{b}$. Then $\vec{PS} = \mathbf{b}$ and $\vec{SR} = \mathbf{a}$. Therefore applying the triangle law to the two separate triangles PQR andPSR we have $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$.

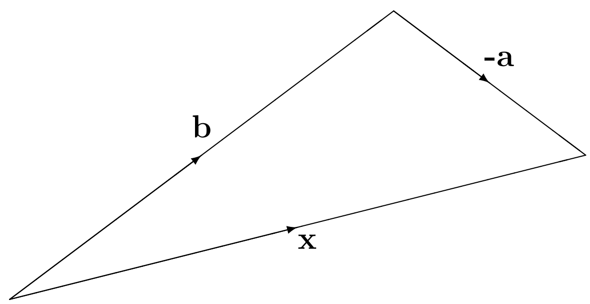
ii) the associative law of addition



We can now add any number of vectors in any order we like to get the same result. $\vec{PR} = \mathbf{a} + (\mathbf{b}+\mathbf{c}) = (\mathbf{a}+\mathbf{b}) + \mathbf{c}$

iii) subtraction

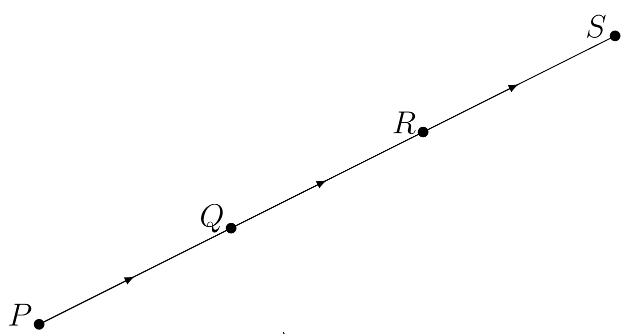
having already defined $-\mathbf{a}$ (negative $\mathbf{a}$) it seems natural to define $\mathbf{b}-\mathbf{a}$ to be $\mathbf{b}+(-\mathbf{a})$. Denoting this by $\mathbf{x}$ we see that $\mathbf{b}+(-\mathbf{a}) = \mathbf{x}$ is equivalent to $\mathbf{x}+\mathbf{a} = \mathbf{b}$.



So $\mathbf{b} - \mathbf{a}$ is the solution $\mathbf{x}$ of the equation $\mathbf{x} + \mathbf{a} = \mathbf{b}$.

Multiplication by a number

In ordinary algebra when we meet an expression like $x + x + x + y + y$ we abbreviate it to $3x + 2y$. So with vectors we abbreviate $\mathbf{a} + \mathbf{a} + \mathbf{a}$ to $3\mathbf{a}$.
In fact if $\mathbf{a} = \vec{PQ} = \vec{QR} = \vec{RS}$ then $\vec{PS} = 3\mathbf{a}$.



In fact we see that $\vec{PS}$ is a vector with 3 times the magnitude and the same sense and direction as $\mathbf{a}$. Similarly if we interpret $-3\mathbf{a}$ as $(-\mathbf{a})+(-\mathbf{a})+(-\mathbf{a})$ then from the same diagram we see that $-3\mathbf{a} = \vec{SP}$. We therefore adopt the following definition.
Given $\mathbf{a}$ and any real number k, we define $k\mathbf{a}$ as follows

1) $|k\mathbf{a}| = k|\mathbf{a}|$
2) The direction of $k\mathbf{a}$ is that of $\mathbf{a}$ if $k > 0$.
3) The direction of $k\mathbf{a}$ is opposite that of $\mathbf{a}$ if $k < 0$.
4) If $k = 0$ we define $k\mathbf{a} = \mathbf{0}$.

Distributive laws

i) $(k+l)\mathbf{a} = (k\mathbf{a}) + (l\mathbf{a})$

ii) $k(\mathbf{a}+\mathbf{b}) = (k\mathbf{a}) + (k\mathbf{b})$

iii) $k(l\mathbf{a}) = (kl)\mathbf{a}$ for all k, l, $\mathbf{a}$, $\mathbf{b}$.

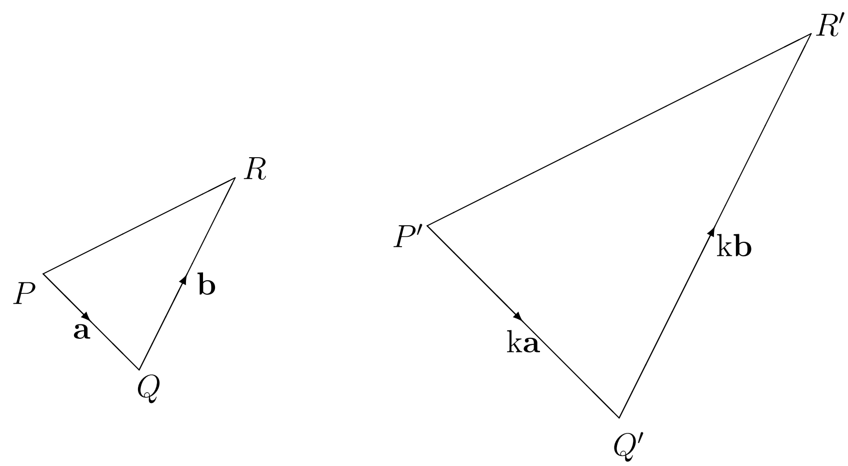
i) suppose k, l are both > 0 and $\mathbf{a} \neq \mathbf{0}$.

then $|(k+l)\mathbf{a}| = |k+l||\mathbf{a}| = (k+l)|\mathbf{a}| = k|\mathbf{a}| + l|\mathbf{a}| = |k\mathbf{a}| + |l\mathbf{a}|$

Furthermore $k\mathbf{a}$, $l\mathbf{a}$ and $(k+l)\mathbf{a}$ all have the same direction and sense. Thus $(k+l)\mathbf{a} = k\mathbf{a} + l\mathbf{a}$.

There are numerous other cases to consider, with k, l either or both negative or zero.

ii)



Let PQR be a triangle with $\vec{PQ} = \mathbf{a}$ $\vec{QR} = \mathbf{b}$ so $\vec{PR} = \mathbf{a} + \mathbf{b}$.

Let $P'Q'R'$ be a triangle with sides parallel to PQR and

$\vec{P'Q'} = k\mathbf{a}$ $\vec{Q'R'} = k\mathbf{b}$ then $\vec{P'R'} = k\mathbf{a} + k\mathbf{b}$.

But the triangles PQR and $P'Q'R'$ are similar. So $|\vec{PR}| = |k||\vec{PR}|$ and so $\vec{P'R'} = k\vec{PR}$ thus $k\mathbf{a} + k\mathbf{b} = k(\mathbf{a}+\mathbf{b})$.

This does not cover the cases where k, $\mathbf{a}$ or $\mathbf{b}$ is zero, or where $\mathbf{a}$ and $\mathbf{b}$ are parallel.

iii) left and right-hand sides both represent vectors with the same magnitude, direction and sense and so are the same.

Summary

The system V of vectors with the system R of real numbers have operations of addition and multiplication by a real number defined with the following properties

a) for all $\mathbf{a}, \mathbf{b} \epsilon V \quad \mathbf{a}+\mathbf{b} \epsilon V \quad$ (Closure)

b) for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \epsilon V \quad (\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c}) \quad$ (Associativity)

c) there exists $\mathbf{0} \epsilon V$, for all $\mathbf{a} \epsilon V \quad \mathbf{a}+\mathbf{0}=\mathbf{a} \quad$ ($\mathbf{0}$ is an identity)

d) for all $\mathbf{a} \epsilon V$ there exists $\mathbf{b} \epsilon V \quad \mathbf{a}+\mathbf{b}=\mathbf{0} \quad$ ($\mathbf{b}$ is an inverse for $\mathbf{a}$ ($\mathbf{b}$=-$\mathbf{a}$)

e) for all $\mathbf{a}, \mathbf{b} \epsilon V \quad \mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a} \quad$ (Commutativity)

f) for all $k \epsilon \mathbf{R}$, for all$\mathbf{a} \epsilon V \quad k\mathbf{a} \epsilon V$

g) for all $k, l \epsilon \mathbf{R}$, for all $\mathbf{a} \epsilon V \quad k(l\mathbf{a})=(kl)\mathbf{a}$

h) for all $\mathbf{a} \epsilon V \quad 1 * \mathbf{a}=\mathbf{a}$

i) for all $k \epsilon \mathbf{R}$, for all $\mathbf{a}, \mathbf{b} \epsilon V \quad k(\mathbf{a}+\mathbf{b})=(k\mathbf{a})+(k\mathbf{b})$ (distributive law)

j) for all $k, l \epsilon \mathbf{R}$, for all $\mathbf{a} \epsilon V \quad (k+l)\mathbf{a}=(k\mathbf{a})+(l\mathbf{a})$ (distributive law)

Notice that we did not include h) in our definition above. We haven't included $0 * \mathbf{a}=\mathbf{a}$. In fact we can deduce this algebraically from the rules above as follows.

$k\mathbf{a}=(0+k)\mathbf{a}=0\mathbf{a}+k\mathbf{a}$ by j)

add $-(k\mathbf{a})$ to both sides

$$\begin{aligned} k\mathbf{a}+-(k\mathbf{a}) &= (0\mathbf{a}+k\mathbf{a})+-(k\mathbf{a}) \\ k\mathbf{a}+-(k\mathbf{a}) &= 0\mathbf{a}+(k\mathbf{a})+-(k\mathbf{a}) && \text{by b)} \\ \mathbf{0} &= 0\mathbf{a}+\mathbf{0} && \text{by d)} \\ \mathbf{0} &= 0\mathbf{a} && \text{by c)} \end{aligned}$$

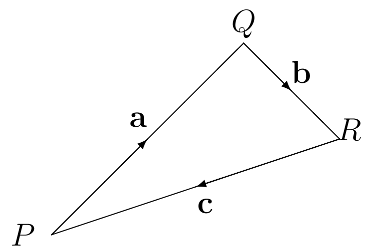
There are many algebraic systems having the above properties. Such a system is a vector space.

Example

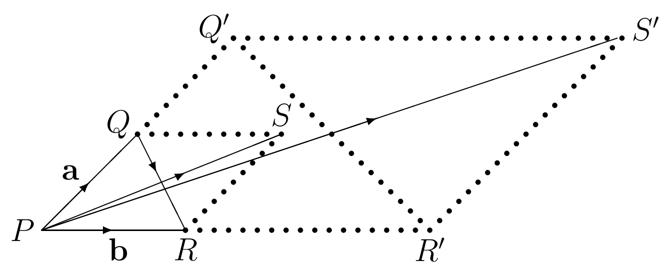
Let V be the set of arithmetic progressions $(a,\ a+d,\ a+2d,\ a+3d,\ ...)$. We define addition term by term, and multiplication by a real number term by term, then V has all the properties (a)-(j). i.e. V is an example of a vector space.

Independence

Consider the three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, where $\mathbf{a} = \vec{PQ},\ \mathbf{b} = \vec{QR},\ \mathbf{c} = \vec{RP}$



then $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, there is a linear relationship between them. However it is possible to choose vectors where there is no such relationship. Consider two vectors $\mathbf{a}$ and $\mathbf{b}$. Choose points PQR so that $\mathbf{a} = \vec{PQ},\ \mathbf{b} = \vec{PR}$.



Let $\vec{PQ'} = k\mathbf{a}$ and $\vec{PR'} = l\mathbf{b}$, then $\vec{PS'} = \mathbf{a} + \mathbf{b}$ and $\vec{PS'} = k\mathbf{a} + l\mathbf{b}$. By the parallelogram rule we can see that S, S' lie in the plane determined by PQR. So if $\mathbf{c}$ is a vector whose direction is not parallel to this plane then $\mathbf{c}$ will not be of the form $k\mathbf{a} + l\mathbf{b}$. $\mathbf{c}$ or any multiple of it cannot be expressed as a linear combination of $\mathbf{a}$ and $\mathbf{b}$. In fact the equation $k\mathbf{a} + l\mathbf{b} + m\mathbf{c} = \mathbf{0}$ will only be possible if $k = l = m = 0$. This leads us to frame the following definitions.

The vectors $\mathbf{a}_1$, ..., $\mathbf{a}_n$ are linearly independent if $k_1\mathbf{a}_1 + k_2\mathbf{a}_2 + ... + k_n\mathbf{a}_n = \mathbf{0}$ only holds with $k_1 = k_2 = ... = k_n = 0$.

If there is a solution with some of the k's non zero then the vectors are linearly dependent. If for example $k_1 \neq 0$ then we can write

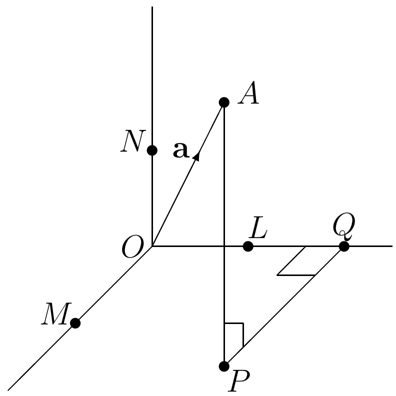
$\mathbf{a}_1 = -\frac{k_2}{k_1}\mathbf{a}_2 - ... - \frac{k_n}{k_1}\mathbf{a}_n = m_2\mathbf{a}_2 + ... + m_n\mathbf{a}_n$.

$\mathbf{a}_1$ is a linear combination of the vectors $\mathbf{a}_2$, ..., $\mathbf{a}_n$. What we are interested

in is finding a set of vectors having the property that all other vectors can be obtained as linear combinations of these vectors. We would naturally look for a set of independent vectors with this property. Many vector spaces have a finite set with this property, that they generate the whole space through linear combinations. Such a set is called a basic set, or a basis. It will not be unique, but it can be proved that all bases contain the same number of vectors. This number is called the dimension of the space.
For example, the set of AP's has dimension 2, with basis
$(1, 1, 1, 1, 1, 1, ...)$ $(0, 1, 2, 3, 4, 5, ...)$

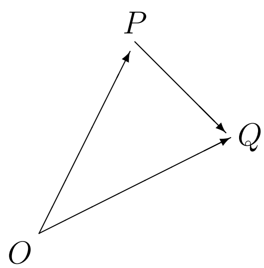
Components

A vector $\mathbf{u}$ is called a unit vector if $|\mathbf{u}| = 1$. If $\mathbf{a}$ is any non-zero vector then there are two unit vectors associated with $\mathbf{a}$, $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$ and $-\hat{\mathbf{a}} = -\dfrac{\mathbf{a}}{|\mathbf{a}|}$.
Let $\mathbf{u} = \vec{OP}$ be a unit vector, and let Π be a plane through 0 perpendicular to $\mathbf{u}$. Let $\mathbf{a}$ be an arbitrary vector not parallel to Π and let $\mathbf{a} = \vec{OA}$ (so A is not in Π). Choose a point Q in Π such that QA is parallel to $\mathbf{u}$, so $\vec{QA} = a_1\mathbf{u}$ for some number a_1.
Then $\vec{OA} = \vec{OQ} + \vec{QA}$
so $\mathbf{a} = \mathbf{q} + a_1\mathbf{u} \qquad (\mathbf{q} = \vec{OQ})$
This gives a decomposition for $\mathbf{a}$. With a fixed unit vector $\mathbf{u}$ this decomposition is unique, for if
$\mathbf{a} = a_1\mathbf{u} + \mathbf{q} = a_1'\mathbf{u} + \mathbf{q}'$
then $(a_1 - a_1')\mathbf{u} = \mathbf{q}' - \mathbf{q}$
The LHS is in the direction of $\mathbf{u}$ and the RHS is perpendicular $\mathbf{u}$, unless both are zero, so $a_1 = a_1'$ and $\mathbf{q} = \mathbf{q}'$.
The number a_1 is called the component of $\mathbf{a}$ in the direction of $\mathbf{u}$.
Now choose three unit vectors at right angles and call them $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$.
Write $\mathbf{u}_1 = \vec{OL}$, $\mathbf{u}_2 = \vec{OM}$, $\mathbf{u}_3 = \vec{ON}$.
Given an arbitrary vector $\mathbf{a} = OA$
we have $\vec{OA} = \vec{OQ} + \vec{QP} + \vec{PA} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$.

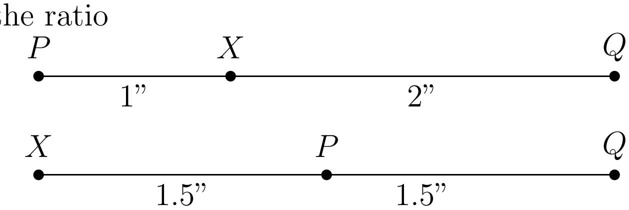


Again the expression is unique, for $a_2\mathbf{u}_2 + a_3\mathbf{u}_3$ is in the plane OMN.
This means that $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ is a basis, and a_1, a_2, a_3 are called the components of $\mathbf{a}$ (relative to the basis $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$)
We can write $\mathbf{a} = (a_1,\ a_2,\ a_3)$
Suppose now we have vectors
$\mathbf{a} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$ $\qquad$ $\mathbf{b} = b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + b_3\mathbf{u}_3$
Then by the laws of vector algebra.
$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{u}_1 + (a_2 + b_2)\mathbf{u}_2 + (a_3 + b_3)\mathbf{u}_3$
$k\mathbf{a} = (ka_1)\mathbf{u}_1 + (ka_2)\mathbf{u}_2 + (ka_3)\mathbf{u}_3$
i.e. $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$
$k(a_1, a_2, a_3) = (ka_1, ka_2, ka_3)$
Now it can be verified that the set of all triples with operations defined above is a vector space. It is therefore isomorphic to the space of displacement vectors.
Since the three axes are orthogonal, Pythagoras' theorem gives
$OA^2 = OQ^2 + QP^2 + PA^2$
$|\mathbf{a}|^2 = a_1^2 + a_2^2 + a_3^2$
In place of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ a notation widely used in three dimensions for three orthogonal unit vectors is $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and instead of $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$, we commonly use $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. These notations have the disadvantage that they do not extend to more than 3 dimensions, but in this course we shall be working in 3 dimensions.
Notice that since the components of a vector are uniquely specified, the equation $\mathbf{a}_1 = \mathbf{a}_2$ is equivalent to $x_1 = x_2, y_1 = y_2, z_1 = z_2$. i.e. in 3 dimensions a vector equation is equivalent to three numerical equations.

Position Vector
Let O be a point fixed in space. The displacement vector $\vec{OP}$ is called the position vector of the point P (relative to O). If in addition to O we have three orthogonal unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the standard notation for a position vector is $\mathbf{r}$ so if P has co-ordinates (x, y, z) we have $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
We can now combine this idea with that of independence to give us a parametric equation for the plane. We have seen that if S' is an arbitrary point in the plane PQR its position vector $\mathbf{r}$ can be expressed as $\mathbf{r} = \vec{OP} + k\vec{PQ} + l\vec{PR}$ for some values of the parameters k, l.
In these calculations we use the triangle law in the form



given $\vec{OP}$ and $\vec{OQ}$, since $\vec{OP} + \vec{PQ} = \vec{OQ}$
we have $\vec{PQ} = \vec{OQ} - \vec{OP}$
Example
Find the parametric equation for the plane through the points
$P(1, 2, 3),\ Q(3, 4, 5),\ R(-1, -2, 4)$
$\vec{PQ} = (2, 2, 2),\ \vec{PR} = (-2, -4, 1)$ so the parametric equations are
$\mathbf{r} = (1, 2, 3) + k(2, 2, 2) + l(-2, -4, 1)$
e.g. $k = 2$, $l = -1$ gives
$\mathbf{r} = (1, 2, 3) + (4, 4, 4) + (2, 4, -1) = (7, 10, 6)$ a point in the plane.
The Ratio Theorem
Suppose we have fixed points P, Q, and a point X which lies on the line PQ and which satisfies $PX : XQ = k : l$
Notice that to specify X we use a sign convention so that X is specified by the ratio



$PX : XQ = 1 : 2$

$PX : XQ = -1 : 2$

Now $\vec{PX} = \frac{k}{l}\vec{XQ}$ and so if P, X, Q have position vectors $\mathbf{p}$, $\mathbf{x}$, $\mathbf{q}$ relative to some O, then

$$\begin{aligned} l(\mathbf{x}-\mathbf{p}) &= k(\mathbf{q}-\mathbf{x}) \\ (k+l)\mathbf{x} &= k\mathbf{q}+l\mathbf{p} \\ \mathbf{x} &= \frac{k\mathbf{q}+l\mathbf{p}}{k+l} \end{aligned}$$

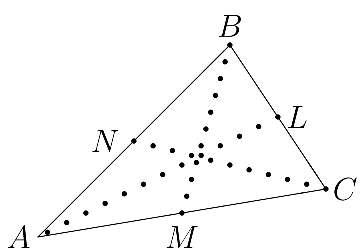
In particular if $k = l = 1$ then the position vector of the mid point of PQ is $\mathbf{x} = \dfrac{1}{2}(\mathbf{p}+\mathbf{q})$

Examples

i) The centoid of a triangle.

Let A, B, C, L, M, N have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$.

Then $\mathbf{l} = \frac{1}{2}(\mathbf{b}+\mathbf{c})$, $\mathbf{m} = \frac{1}{2}(\mathbf{c}+\mathbf{a})$, $\mathbf{n} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$



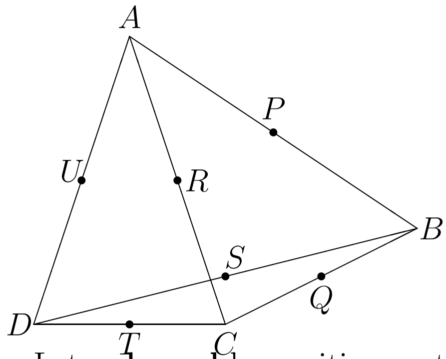
Consider the point G on AL such that $AG : GL = 2 : 1$. This has position vector

$\frac{1}{3}(\mathbf{a}+2\mathbf{l}) = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$

also $\frac{1}{3}(\mathbf{b}+2\mathbf{m}) = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}) = \frac{1}{3}(\mathbf{c}+2\mathbf{n})$

Thus the point G lies on AL, BM, CN. It is called the centroid of the triangle. In general if we have points $\mathbf{a}_1, \ldots, \mathbf{a}_n$ we define their centroid to be $\frac{1}{n}(\mathbf{a}_1 + \ldots + \mathbf{a}_n)$.

ii)



Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be position vectors of A, B, C, D.

$\mathbf{p} = \frac{1}{2}(\mathbf{a}+\mathbf{b}) \qquad \mathbf{t} = \frac{1}{2}(\mathbf{c}+\mathbf{d})$

The mid-point of PT has position vector $\frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})$. By symmetry (or verification) this is also the mid-point of QU and RS. So the lines joining the mid-points of opposite sides of a tetrahedron are concurrent at the centroid.

Equations of a line

i) Suppose we are given two points $\mathbf{p}$ and $\mathbf{q}$ and we want the equation of a line through P and Q. If R is an arbitrary point on the line then for some k, l we have

$\mathbf{r} = \dfrac{k\mathbf{q}+l\mathbf{p}}{k+l} = \dfrac{k}{k+l}\mathbf{q} + \dfrac{l}{k+l}\mathbf{p}$

(Note the convention of $\mathbf{r}$ for the position vector of a variable point.)

Let $\dfrac{k}{k+l} = t$ then $\dfrac{l}{k+l} = (1-t)$.

So $\mathbf{r} = t\mathbf{q} + (1-t)\mathbf{p} \quad t \epsilon \mathbf{R}$

Note that $0 < t < 1$ gives the region between P and Q.

ii) If O is the origin and $\vec{OP} = \mathbf{p}$, $\vec{OQ} = \mathbf{q}$ then from the triangle law $\vec{PQ} = \mathbf{q} - \mathbf{p}$.

Given

$$\begin{aligned}\mathbf{r} &= t\mathbf{q} + (1-t)\mathbf{p} \\ \mathbf{r} &= \mathbf{p} + t(\mathbf{q}-\mathbf{p})\end{aligned}$$

If we are given a point A with position vector $\mathbf{a}$ and a vector $\mathbf{b}$ then the line through A in direction $\mathbf{b}$ has equation $\mathbf{r} = \mathbf{a} + t\mathbf{b} \quad t\epsilon\mathbf{R}$.

iii) Three points A, B, C are collinear. So C lies on the line through A and B, so for some $t\epsilon\mathbf{R}$

$\mathbf{c} = t\mathbf{a} + (1-t)\mathbf{b}$

$t\mathbf{a} + (1-t)\mathbf{b} - \mathbf{c} = \mathbf{0}$

Notice that the sum of the coefficients is zero.

Now suppose $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = \mathbf{0}$, with $\alpha + \beta + \gamma = 0$, and not all three zero.

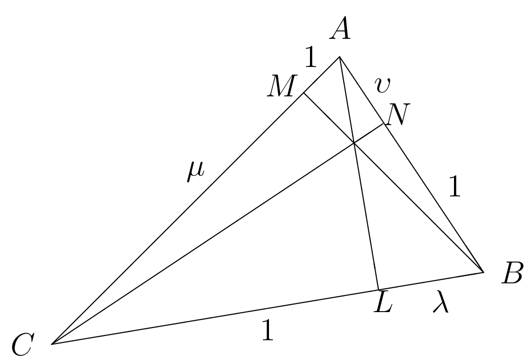
Suppose without loss of generality that $\gamma \neq 0$, then $\frac{\alpha}{\gamma}\mathbf{a} + \frac{\beta}{\gamma}\mathbf{b} + \mathbf{c} = 0$

so $c = -\frac{\alpha}{\gamma}\mathbf{a} - \frac{\beta}{\gamma}\mathbf{b}$.

Let $-\frac{\alpha}{\gamma} = t$ then $-\frac{\beta}{\gamma} = 1 - t$ since $\alpha + \beta + \gamma = 0$.

i.e. A, B, and C are collinear iff $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = \mathbf{0}$, with $\alpha + \beta + \gamma = 0$ and not all three are zero.

Example



Suppose $BL : LC = \lambda : 1 \qquad CM : MA = \mu : 1 \qquad AN : NB = \upsilon : 1$
In many problems the algebra is simplified by choosing a special origin. In this case choose C to be the origin and let the position vectors of A and B be $\mathbf{a}$ and $\mathbf{b}$. Then

$M = m\mathbf{a}$ where $m = \dfrac{\mu}{\mu+1}$ $\qquad L = l\mathbf{b}$ where $l = \dfrac{1}{\lambda+1}$

$N = \dfrac{\upsilon\mathbf{b}+\mathbf{a}}{\upsilon+1}$

Let $\mathbf{r}$ be the position vector of the point R where AL meets BM.

Since R lies on AL, $\quad \mathbf{r} = \mathbf{a} + k(l\mathbf{b} - \mathbf{a}) \qquad$ for some $k\epsilon\mathbf{R}$

Since R lies on MB, $\quad \mathbf{r} = m\mathbf{a} + h(\mathbf{b} - m\mathbf{a}) \qquad$ for some $h\epsilon\mathbf{R}$

So $\mathbf{a} + k(l\mathbf{b} - \mathbf{a}) = m\mathbf{a} + h(\mathbf{b} - m\mathbf{a})$

$\mathbf{a}(1 - k - m + mh) = \mathbf{b}(h - kl)$

Now $\mathbf{a}$ and $\mathbf{b}$ are non-zero and not parallel, so $h = kl$ and $1 - k = m(1 - h)$.

So $1 - k = m - mkl \quad$ i.e. $k = \dfrac{1-m}{1-lm} \quad lm \neq 1$

Thus $\mathbf{r} = \mathbf{a} + \dfrac{1-m}{1-lm}(l\mathbf{b} - \mathbf{a}) = \dfrac{m(1-l)\mathbf{a} + l(1-m)\mathbf{b}}{1-lm}$

Now $\mathbf{n} = \frac{\upsilon\mathbf{b}+\mathbf{a}}{\upsilon+1}$ and CN passes through R iff $\mathbf{n} = p\mathbf{r}$ for some $p\epsilon\mathbf{R}$

So $\dfrac{1}{\upsilon+1}\mathbf{a} + \upsilon\upsilon + 1\mathbf{b} = p\left(\dfrac{m(1-l)}{1-lm}\mathbf{a} + \dfrac{l(1-m)}{1-lm}\mathbf{b}\right)$

Now compare coefficients and eliminate p

$\iff \upsilon = \dfrac{l(1-m)}{m(1-l)} \quad 1 - m = \dfrac{1}{\mu+1} \quad 1 - l = \dfrac{\lambda}{\lambda+1}$

$\iff \upsilon = \dfrac{1}{\lambda+1}\dfrac{1}{\mu+1}\dfrac{\mu+1}{\mu}\dfrac{\lambda+1}{lambda}$

$\iff \lambda\mu\upsilon = 1$

This result is know as Ceva's Theorem, and is often written as

$\dfrac{BL}{LC}\dfrac{CM}{MA}\dfrac{AN}{NB} = 1$

Notice that the configuration can also appear in the form below