SOES 6006: Climate Dynamics
Worksheet for Workshop Session on the Basic Mathematics of Linear Dynamical Systems

with answers

Important Notes

A. ***You should ideally be able to work on quantitative problems using any system of notation and any system of units you wish (or are asked to use). While some systems of notation and units (e.g. SI units) are commonly used and are more or less standard, this is only for convenience and not of great importance. In this course you will encounter inconsistencies of notation and unfamiliar and peculiar units. We make no apology for this. That's the way it is, in real life, and one must learn to live with it…***

B. ***When working on quantitative problems, you should work in symbolic notation to the maximum extent possible*** (we didn't in the previous exercise, but that was only to introduce the concepts) ***for as long as possible, and only substitute numerical values at the very end. In that way you will (a) obtain a generally applicable result; (b) be able to profit from any cancellation and simplification which may be possible; (c) be able to check for symmetry and dimensional consistency, and so trap and avoid many errors…***

1) Derive the first order linear differential equation for the evolution of the concentration C of some substance, in a reservoir of volume V, if a fraction λ of the inventory is lost per unit time, and the substance is generated by a source term S units per unit volume per unit time…

$$\frac{dC}{dt} + \lambda C = S$$

Why does the volume of the reservoir not appear in this equation ?

2) If S is zero, the equation is said to be ***homogeneous*** (if you don't know why, please ask). Does the homogeneous equation have a steady-state solution ? [Hint, set dC/dt to zero, and solve for C]. If so, what is it ?

$$C = 0$$

3) Does the ***inhomogeneous*** equation (with $S > 0$) have a steady-state solution (assuming that S is constant w.r.t. time) ? If so, call this steady-state solution C_S ... What is it ?

$$C_S = S/\lambda$$

4) Write down the differential equation for the transient difference C' between the general solution C and the steady-state solution C_S {i.e. for $C' = C - C_S$}. *[Note: The ′ here just denotes a difference, **not** a differential operator..] [Hint: rewrite this last expression to give an expression for C, substitute this in the differential equation, and collect terms in C' and simplify as far as possible]*

$$\frac{dC}{dt} + \lambda C = S \text{ , but } C = C_S + C' \text{ , so } \frac{dC'}{dt} + \lambda C' = S - \lambda C_S = 0 \text{, and so } \frac{dC'}{dt} + \lambda C' = 0$$

In general the equation for the transient solution is thus always homogeneous.

5) This difference C', between the general solution C and the steady-state solution C_0, describes the ***transient solution*** which may occur as a result of a ***perturbation*** which takes the system away from the steady-state, or from starting at a level different from the steady-state, as a result of the ***initial conditions***.
Write down the most general solution for C' .

[Note: solutions to differential equations cannot usually be derived by a step-by-step process (integration is an inverse process). One normally has to proceed by recognising the general form, and using trial & error, and/or by looking them up in books where the results obtained in that way by clever people are recorded. You need to be able to recognise the simple standard forms of differential equations and their solutions. This one has an exponential solution.] Hint: include any necessary constants of integration, so that you will be able to fit any necessary boundary or initial conditions…

$$C' = A\exp(-\lambda t) + B$$

6) Remembering that C' is the solution of the homogeneous equation for a transient, what is its steady-state solution (when $t \to \infty$) ?

$C'=0$

7) Can you use that result to eliminate or determine any constants of integration ?

Yes, it determines that $B=0$

8) What, therefore, is actually the general solution for C' ?

$$C' = A\exp(-\lambda t)$$

9) What, therefore, is the general solution for C itself ?

$$C = C_S + A\exp(-\lambda t)$$

10) If the system starts with the initial condition $C = C_0$ when $t = 0$, use this initial condition to determine A and hence derive the full specific solution for C

$C = C_S + A\exp(-\lambda t)$, but for $t = 0$, $C = C_0$, and $\exp(-\lambda t) = 1$, so $C_0 = C_S + A$, and so $A = (C_0 - C_S)$

thus in this case $C = C_S + (C_0 - C_S)\exp(-\lambda t)$

Note that this form of the solution [i.e. ***solution =final + (initial – final) × transient****] where the transient solution tends to zero eventually, is generally applicable for well-behaved first-order (and other) initial value problems, so it can be written down immediately, and there is no need to solve it all over again afresh every time such a problem is encountered... Note also that λ defines the rate at which the transient solution decays, and so τ = 1/λ is the characteristic* ***response time*** *for the approach of the system to equilibrium.*

Sketch the solutions for both positive and negative perturbations from the steady-state

11) Is the steady-state solution ***stable ?***

The solution is stable if the result of any perturbation gets progressively smaller (rather than bigger) with time, i.e. if transient term tends to zero as $t \to \infty$*, so*

a) For $\lambda > 0$

Yes

b) For $\lambda < 0$

No

12) Another useful way to check for stability is to try a trial solution of the form $e^{i\omega t}$ in the differential equation for the transient term, i.e. in the equation for the deviation from steady state.

Note that $e^{i\omega t}$ *is the standard complex number form for an oscillatory solution, since* $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ *(often known as de Moivre's theorem) where i is the square root of –1, and only the real part of this complex expression is taken to correspond to reality. [If you don't understand complex numbers, please ask…] ω (in radians per unit time) here denotes the* ***angular frequency*** *ω = 2πf where f is the ordinary frequency (in cycles per unit time, e.g. Hz). It is used mainly just for convenience (to avoid writing 2πf over & over again).*
NB also that for this trial solution we always have $dC'/dt = i\omega C'$*...*

Try this trial solution for the equation for *C'* and comment on the result…

In this case we find that $dC'/dt = i\omega C' = -\lambda C'$ *so that* $\omega = i\lambda$ *and the solution is therefore the decaying exponential* $\exp(-\lambda t)$*, which we knew already so it's not very interesting. However in more realistic and complicated cases the result of this technique can be very useful, especially when there is time-dependent forcing of the source and/or boundary terms…*

13) Consider the case where the system is subjected to periodic forcing (e.g. seasonal, Milankovitch (orbital) etc) via the source term, such that $S = S_0 + Fe^{i\omega t}$, say. Show that we now have the perturbation equation $\frac{dC'}{dt} + \lambda C' = Fe^{i\omega t}$, and use a trial solution $C' = Ae^{i\omega t}$ to find an expression for the system ***Gain*** $G(\omega)$ which expresses the amplitude *A* of the system response relative to the forcing *F*, i.e. $G(\omega)$ =*A/F*, as a function of ω...

If we insert $C' = Ae^{i\omega t}$ *and remember that* $dC'/dt = i\omega C'$ *we obtain* $i\omega Ae^{i\omega t} + \lambda Ae^{i\omega t} = Fe^{i\omega t}$*, so* $(i\omega + \lambda)A = F$ *, and* $G(\omega) = A/F = 1/(\lambda + i\omega)$

14) Sketch the magnitude (modulus) $|G(\omega)|$ as a function of ω. Specifically, how does it behave w.r.t. ω, and what values does it have…

a) For low frequency perturbations, i.e. $\omega << \lambda$?

$G(\omega) \approx 1/\lambda$*, which is a constant,*

NB: the low frequency response (gain) is large if λ is small, and vice versa, so that systems with a slow response rate (i.e. a long response time) are very sensitive (in the long term…)

b) For high frequency perturbations, i.e. $\omega >> \lambda$?

$G(\omega) \approx 1/i\omega$*, so* $|G(\omega)| \approx 1/\omega$*, which is inversely proportional to ω, and therefore not only small (less than the low frequency "DC" response) because* $\omega >> \lambda$*, but also progressively smaller and smaller for higher frequencies.*

This characteristic type of response is called a **low-pass filter** response, because low frequency perturbations are "passed" through the system, by comparison with high frequency perturbations which are progressively attenuated. **High-pass filter** responses are also possible (as are higher order responses with sharper cut-offs, in more complicated systems, and many other interesting things…).

*[Note for mathematicians: this is just a quick & easy way of finding the **frequency response** of a system. This can be done much more generally and rigorously using Fourier, Laplace or Z-transforms. In general the frequency response of a linear system is the Fourier transform of its **impulse response**, i.e. its response to a perturbation in the form of a Dirac **delta-function**].*

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