

Basic Linear System Dynamics

Reservoirs & Fluxes

Steady-states & Transients

Residence & Response times

System Gain & Frequency Response

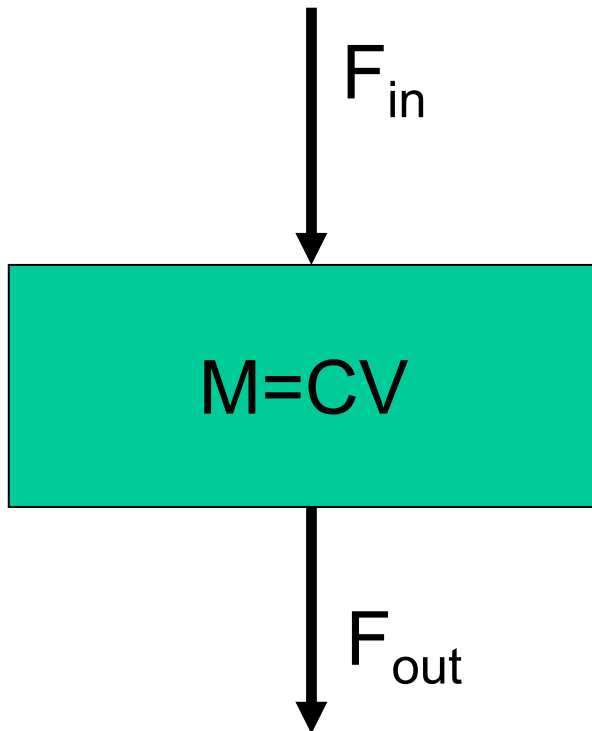
Feedbacks, Loops & Oscillations

Originally by John Shepherd, Modified by Kevin
Oliver, 2010

Basic System Dynamics

- ◆ Most theory is for ***linear*** systems
 - ***Non-linear*** systems are much trickier, but approximations to linear systems can be obtained for *small perturbations*.
- ◆ Concepts: Fluxes and inventories
- ◆ ***Steady-state*** (equilibrium) & ***transient*** (disequilibrium) conditions (*statics vs. dynamics*)
- ◆ Reservoir effects: finite response times
 - Residence times, Transit times, Mixing times, Reaction times, etc
- ◆ Multiple/continuous reservoirs (e.g. the ocean)
 - *Distributions* of response times & ages

Single reservoir residence time & response time (1)



V = volume of reservoir (fixed)

C = concentration (quantity per unit volume)

M = total mass or quantity (= inventory)
(= “standing stock”)

$$M=CV$$

F =flux (quantity per unit time)

NB: may (often) be per unit area

Change = Input - Output

$$\therefore \frac{dM}{dt} = F_{in} - F_{out}$$

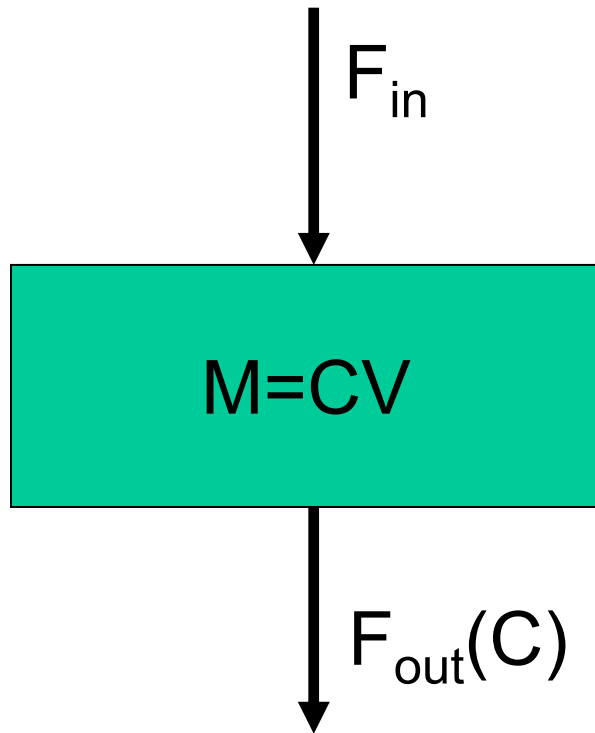
at **steady-state** $\frac{dM}{dt} = 0$

$$\therefore F_{in} = F_{out} \quad \text{and} \quad \tau_{res} = M/F_{in} = M/F_{out}$$

NB: Quantity may be measured as mass, moles, etc, etc...

One **must** ensure dimensional consistency (i.e. the **units** check out OK)

Single reservoir residence time & response time (2)



In general F_{in} and F_{out} may depend on C
e.g. in a **first-order removal rate** process

$$F_{out} = k'C = kCV$$

where $k' = kV$, where $[k] = 1/[\text{time}]$

$$dM/dt = d(CV)/dt = F_{in} - kCV$$

$$dC/dt = S - kC$$

where **source term** $S = (F_{in}/V)$

(quantity per unit volume per unit time)

Solution is **exponential approach to steady-state**

$$C = (S/k) \{1 - \exp(-kt)\}$$

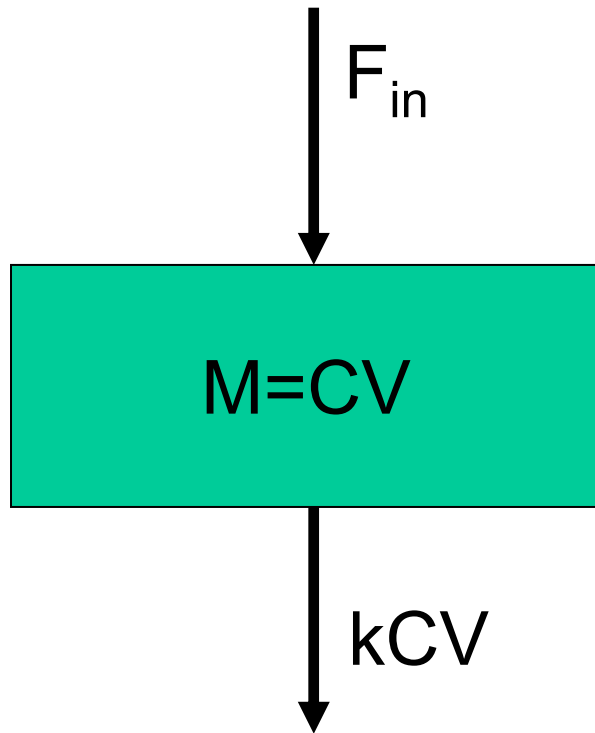
Response time = residence time = $\tau_{res} = 1/k$

NB: (1) for radio-active decay $k = \lambda$ (the radio-active decay constant)

(2) for multiple (**parallel**) removal processes $k_{tot} = k_1 + k_2 + k_3 \dots$

and hence $\tau_{res} = 1 / \{1/\tau_1 + 1/\tau_2 + 1/\tau_3\}$ like “resistances in parallel”
in this case the **fastest** process (with the smallest τ value) dominates

Single reservoir residence time & response time (3)

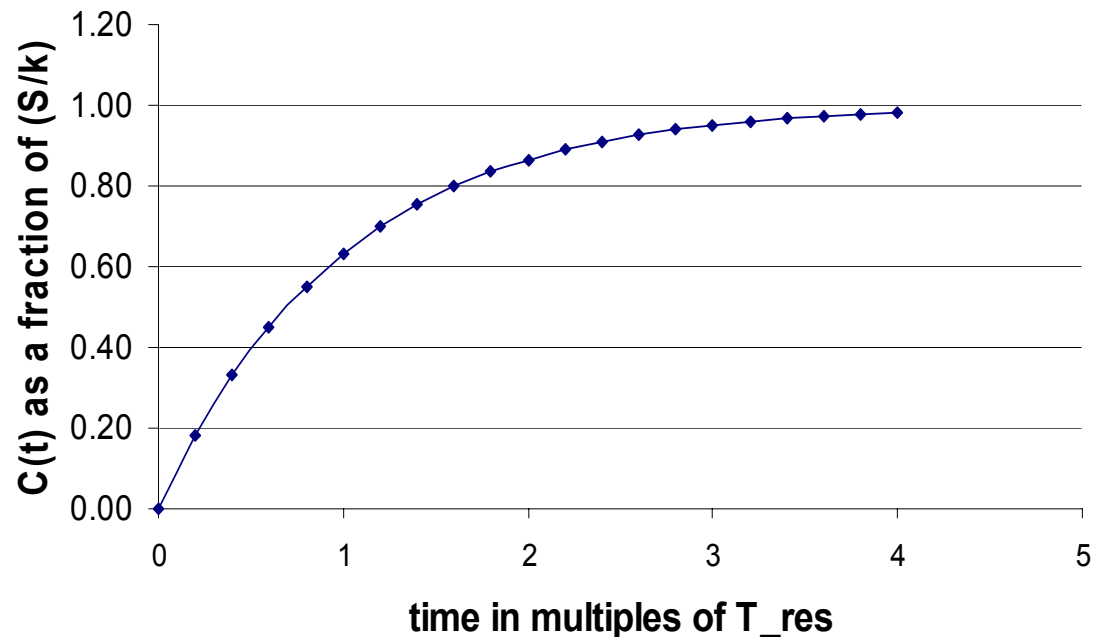


Solution is ***exponential approach to steady-state***

$$C = (S/k) \{1 - \exp(-kt)\}$$

Response time = residence time = $\tau_{res} = 1/k$

Response of a single reservoir



Characteristic system timescales

- ◆ **Residence time;** $\tau_{\text{res}} = 1/k (= 1/\lambda)$
- ◆ **Transit time (advection);** $\tau_{\text{adv}} = L/u$
 - Where L = length scale, u = velocity
- ◆ **Mixing time (diffusion);** $\tau_{\text{mix}} = L^2/2K$
 - Where K = diffusion coefficient
 - Usually due to turbulence
 - May differ (a lot) in different spatial dimensions
 - Horizontal mixing is a lot faster than vertical mixing
 - K values are $\sim 10^8$ times larger in horizontal than in vertical
- ◆ **Scavenging/deposition time;** $\tau_{\text{dep}} = D/v_{\text{dep}}$
 - Where D = depth scale, v_{dep} = deposition velocity
 - Also applies to **gas exchange**; so $\tau_g = D/v_g$
- ◆ **NB: Flux (per unit area) = velocity $\times$ concentration**
 - (with concentration expressed as quantity per unit volume)
- ◆ ***With processes in series, the slowest process dominates***

Multiple reservoirs

“Mass” fluxes and exchanges

- ◆ Consider simple exchanges between reservoirs
- ◆ Exchange time $\tau_{\text{exch}} = \text{Vol}/Q$
 - Where $Q = u_{\text{exch}} \times \text{Area}$
 - Dimensions of Q are [Volume/Time]
 - i.e “mass” or volume flux, e.g. m^3/sec or Sv
 - Q and u_{exch} may be due to advective or diffusive exchange (or both)
- ◆ For multiple reservoirs (e.g. in a box model)
 - For the i^{th} box, with volume V_i , and exchange coefficients Q_{ij} between the i^{th} and the j^{th} boxes

$$\tau_i = V_i / \sum_j Q_{ij}$$

So the exchanges operate in parallel, and the fastest dominates

Basic System Dynamics

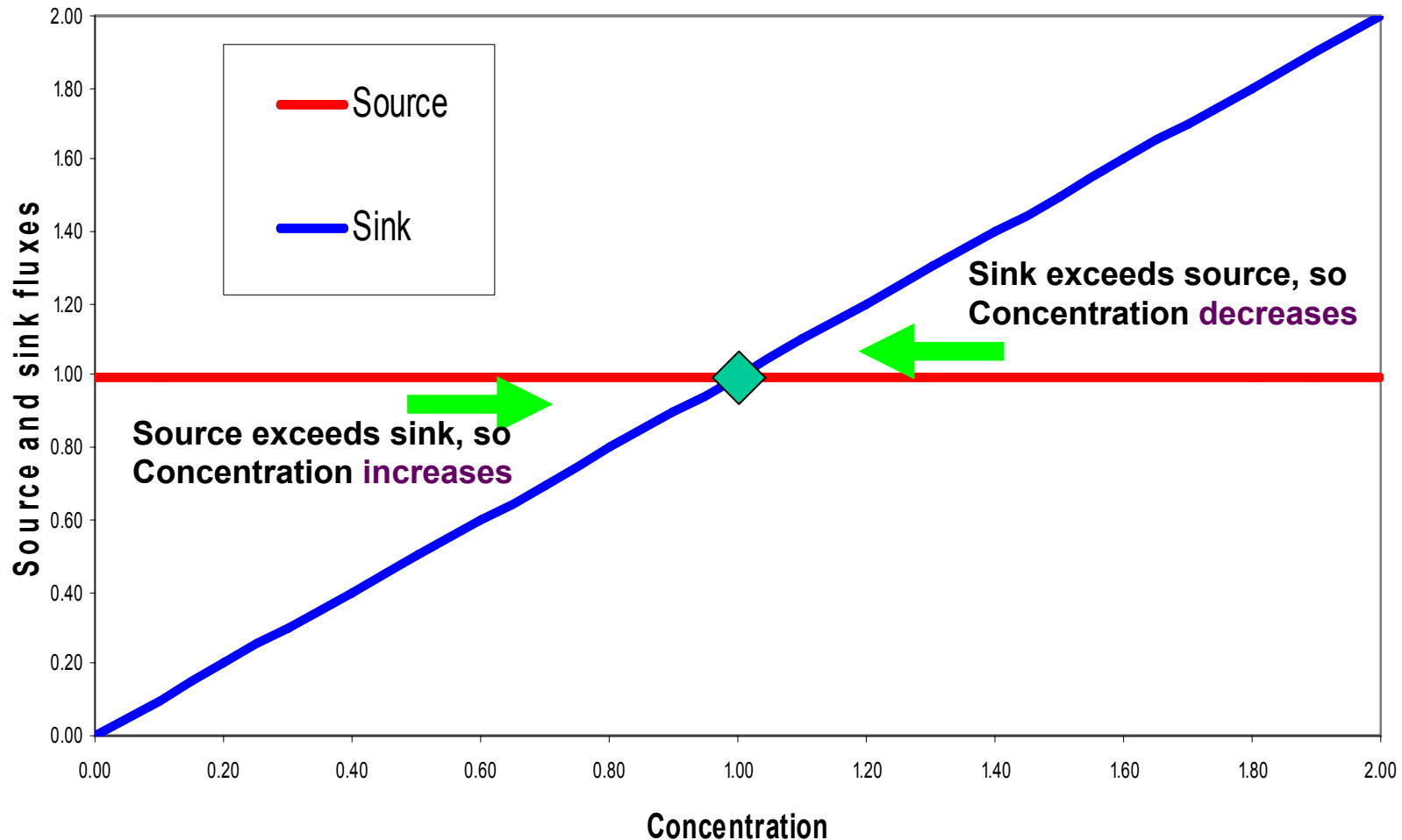
Feedbacks, Loops & Oscillations

Linear System Dynamics: Feedbacks

- ◆ ***Positive*** feedbacks – if these outweigh negative feedbacks:
 - multiple alternative states (separated by **repellors**)
 - hysteresis & rapid transitions (switching)
- ◆ ***Negative*** feedbacks – if these outweigh positive feedbacks:
 - if ***instantaneous*** : stabilisation, stable states (**attractors**)
 - if ***delayed*** : may get resonance (amplification) & possibly oscillations (at characteristic frequencies)

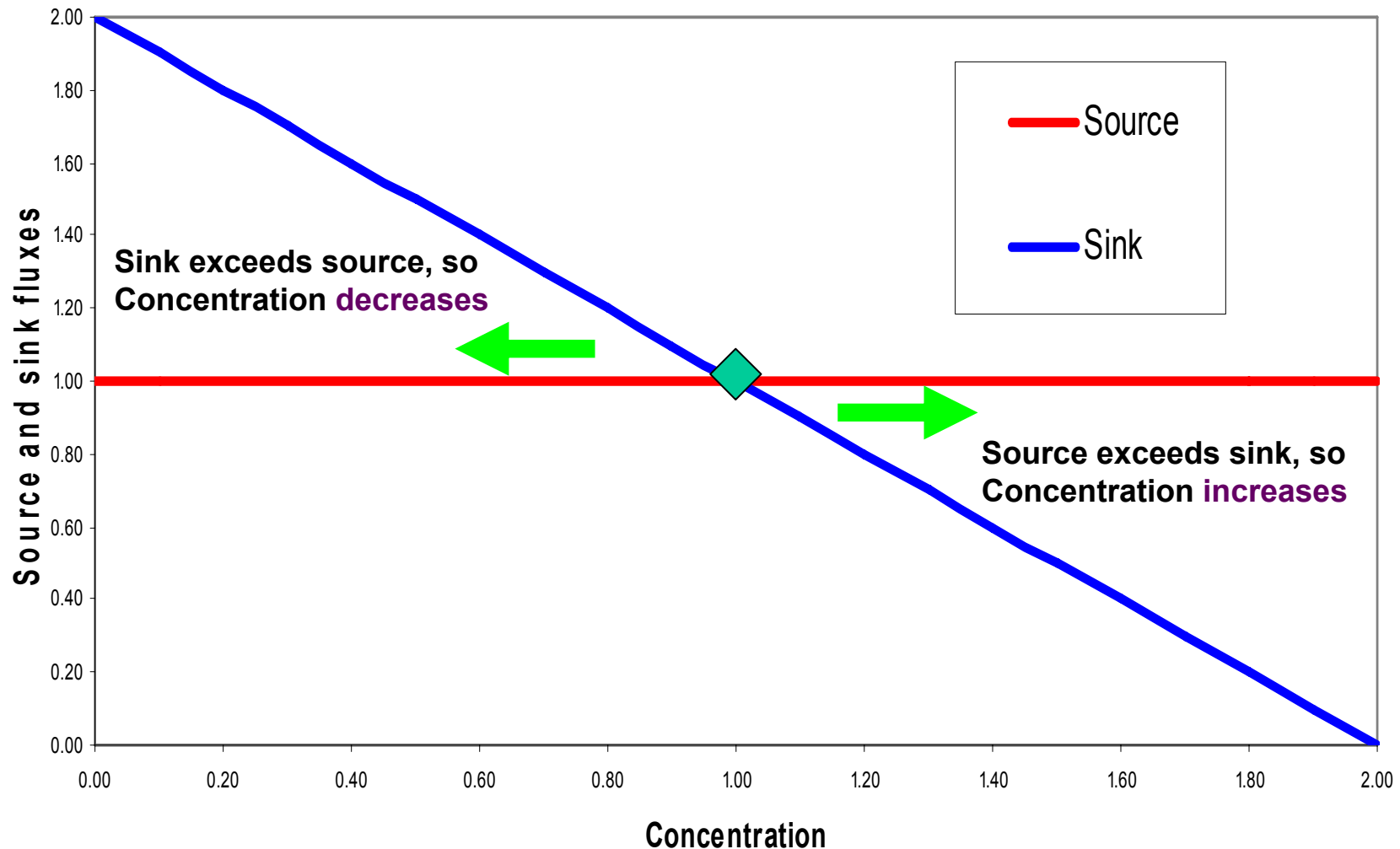
Negative feedback: constant source, but sink flux *increases* with concentration

Balance of source and sink

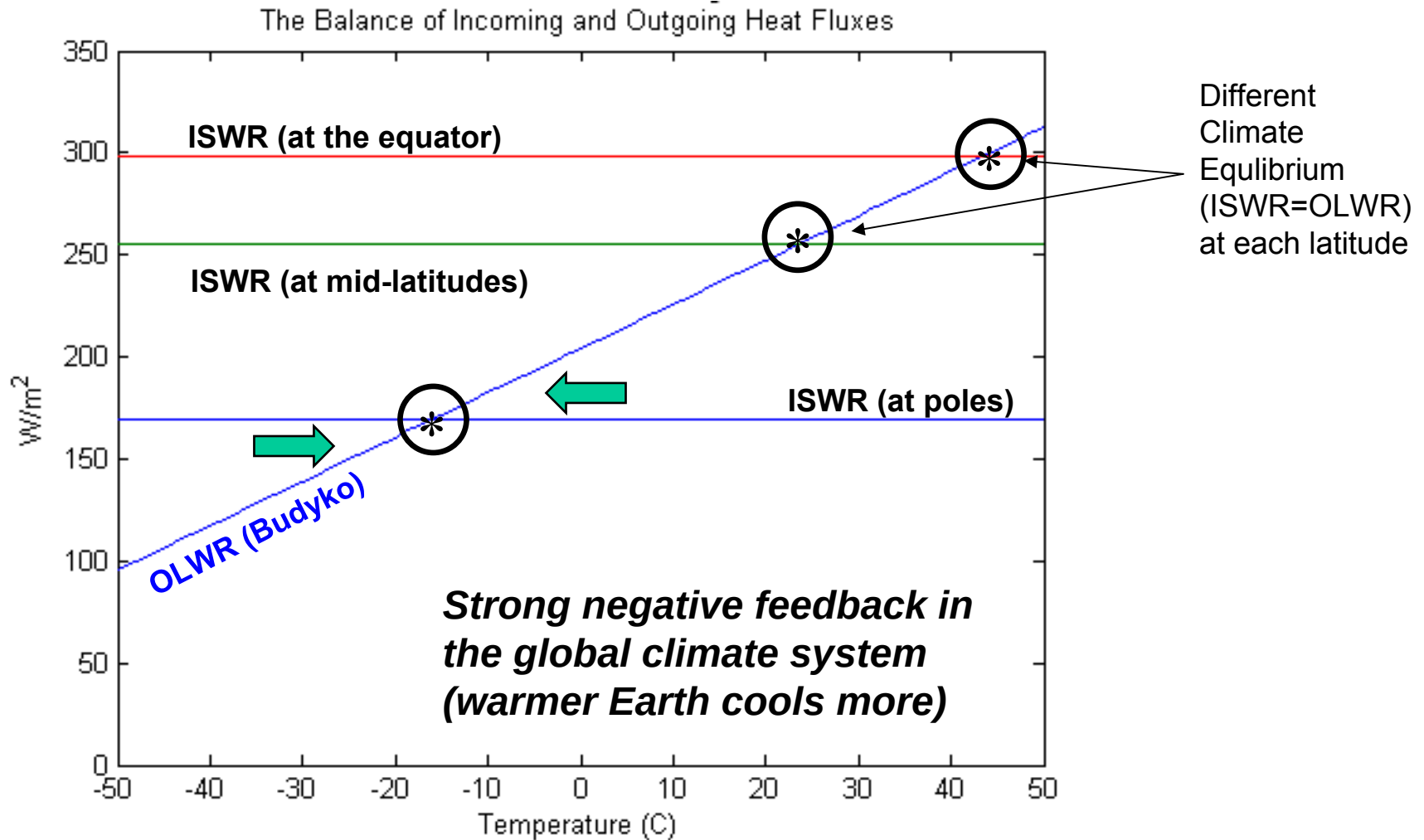


Positive feedback: constant source, but sink flux *decreases* with concentration

Balance of source and sink



Equating ISWR and OLWR (see Lecture 7)

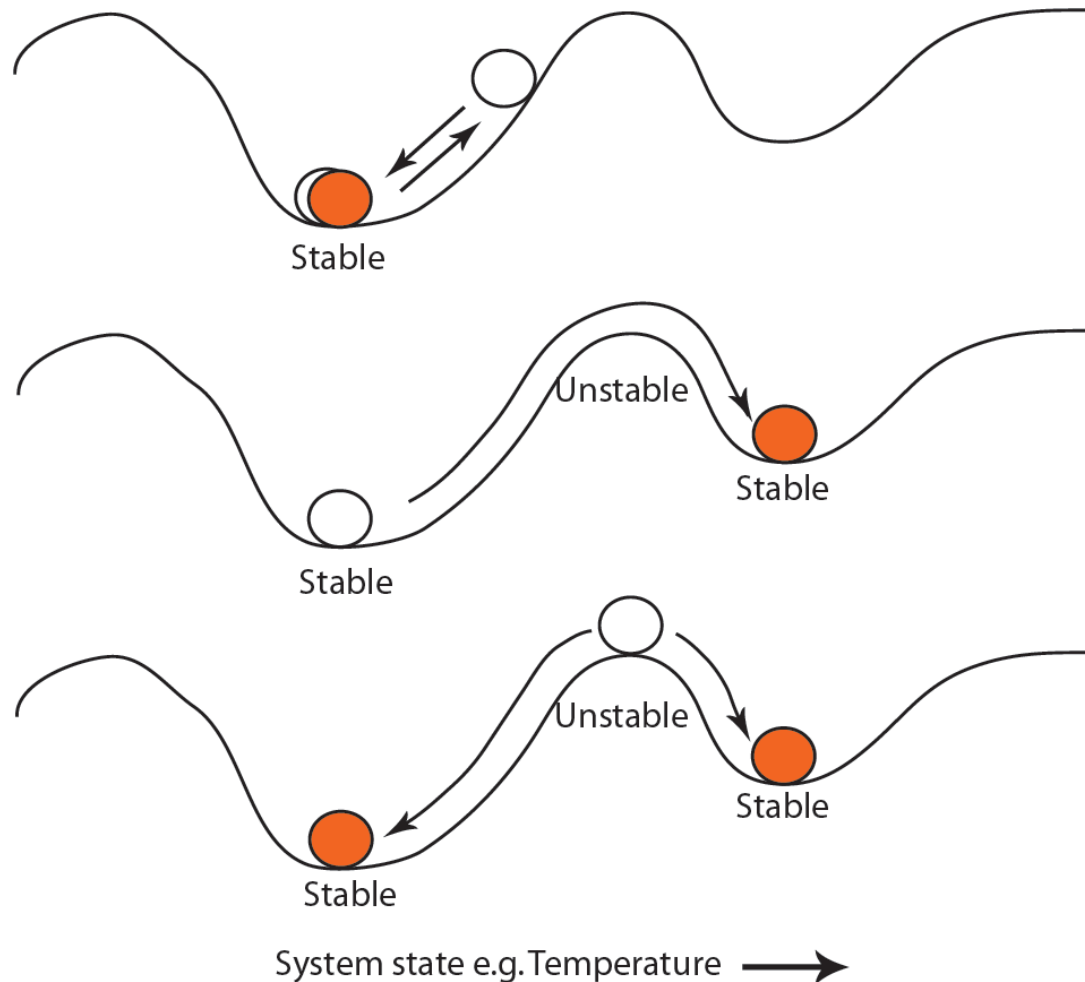


- ◆ If $T < T^*$, $\text{ISWR} > \text{OLWR} \Rightarrow$ warmer
- ◆ If $T > T^*$, $\text{ISWR} < \text{OLWR} \Rightarrow$ cooler

Feedbacks & response to perturbations

- ◆ If the concentration is perturbed from equilibrium...
- ◆ With ***negative*** feedback, the sink term changes so as to reduce the perturbation
 - And so restore the equilibrium
 - $\therefore$ the feedback is ***stabilising***
 - ***The system “seeks” a stable steady-state***
- ◆ With ***positive*** feedback, the sink term changes so as to increase the perturbation
 - And so increase the dis-equilibrium
 - $\therefore$ the feedback is ***destabilising***
 - ***If positive feedbacks outweigh negative feedbacks, the system perturbation grows irreversibly*** (though in practice the linearity of the system usually breaks down instead)

Stable, and unstable equilibria (strictly, *stationary points*) of the system



Negative Feedback

A = Open loop gain

f = feedback fraction

B = Af = Loop gain

G = Overall (closed loop) gain
= Transfer function

$$V_{out} = A V_{in} - (Af) A V_{in} + (Af)^2 A V_{in} - (Af)^3 A V_{in} + (Af)^4 A V_{in} \dots$$

$$V_{out} = A V_{in} (1 - Af + (Af)^2 - (Af)^3 + (Af)^4 - \dots)$$

[i.e., a power series]

$$V_{out} = A V_{in} (1 + Af)^{-1}$$

$$V_{out} (1 + Af) = A V_{in}$$

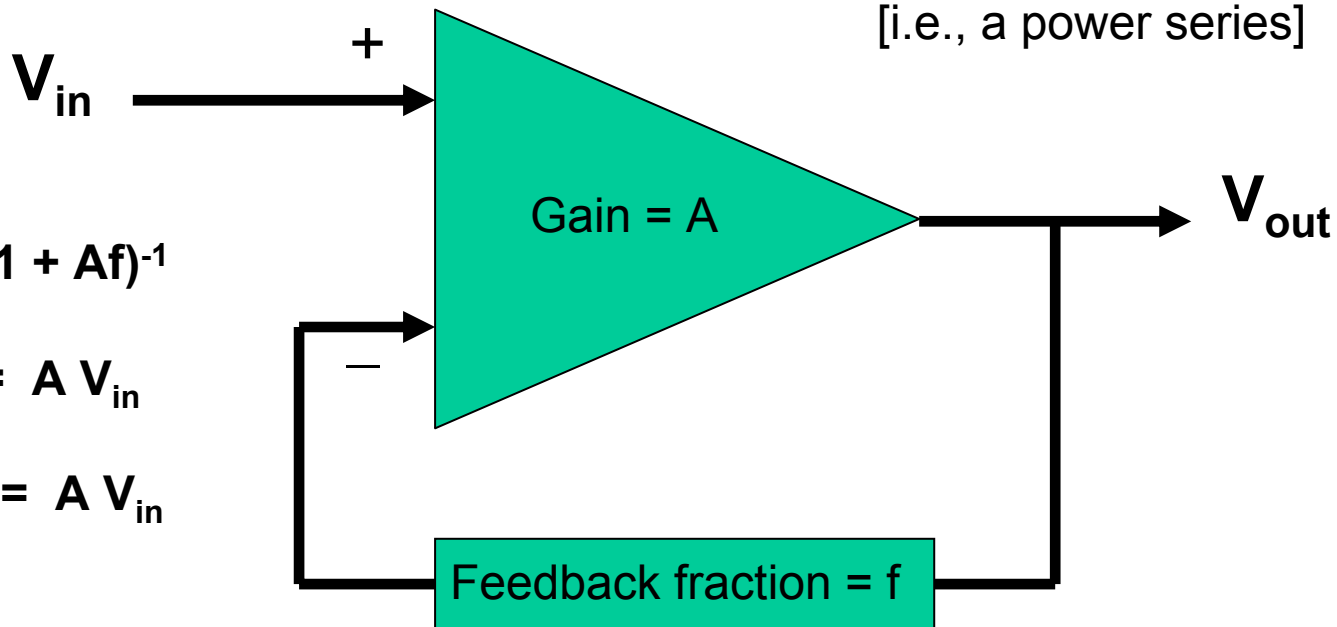
$$V_{out} + Af V_{out} = A V_{in}$$

$$V_{out} = A (V_{in} - f V_{out})$$

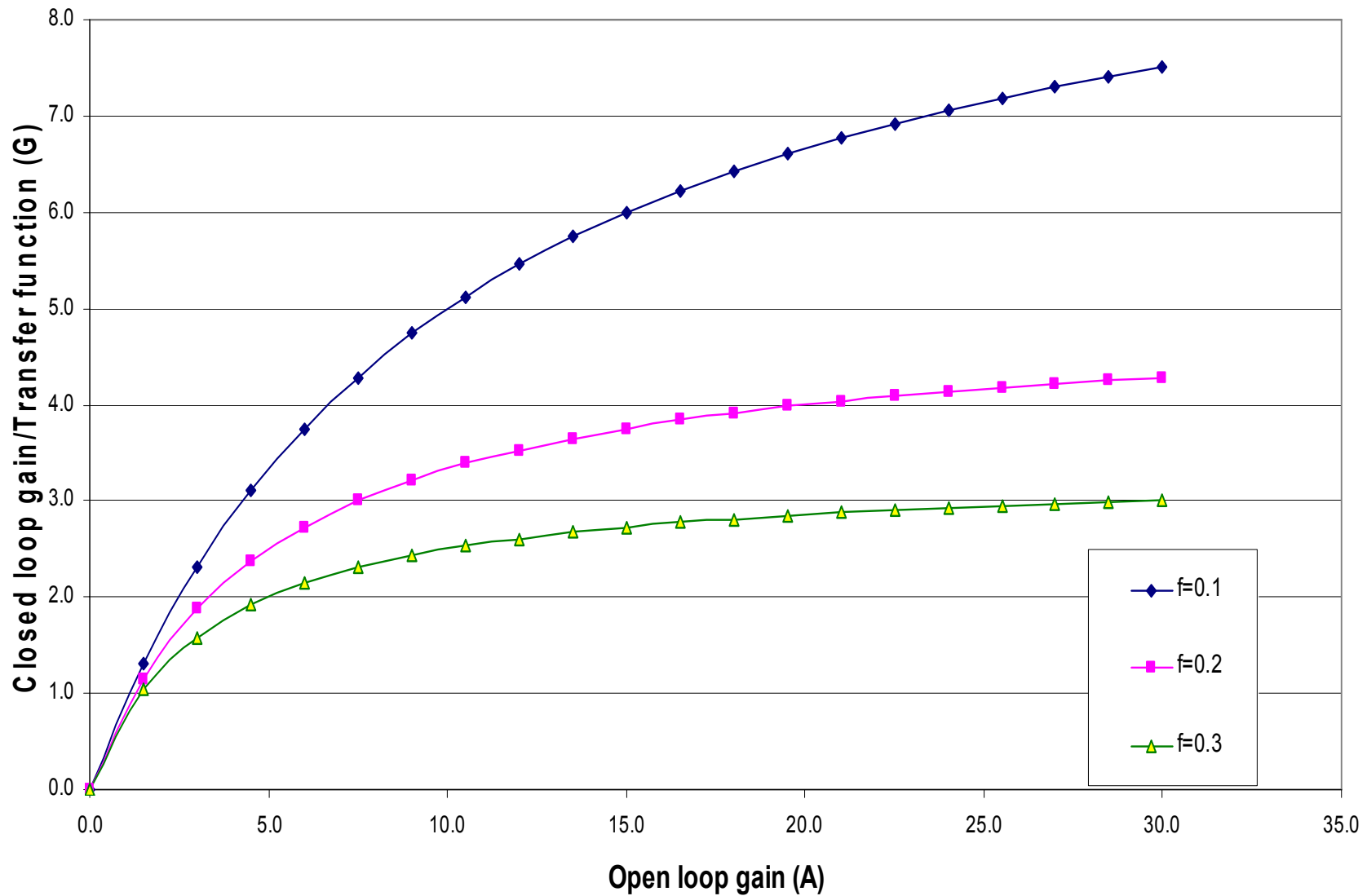
$$\therefore G \rightarrow 1/f \text{ as } A \rightarrow \infty$$

$$\therefore G = V_{out}/V_{in} = A / (1+Af) = A/(1+B)$$

so $G \ll A$, and is “stable” (independent of A)



Negative Feedback: dependence on open loop gain



Positive Feedback

A = Open loop gain

f = feedback fraction

B = Af = Loop gain

G = Overall (closed loop) gain
= Transfer function

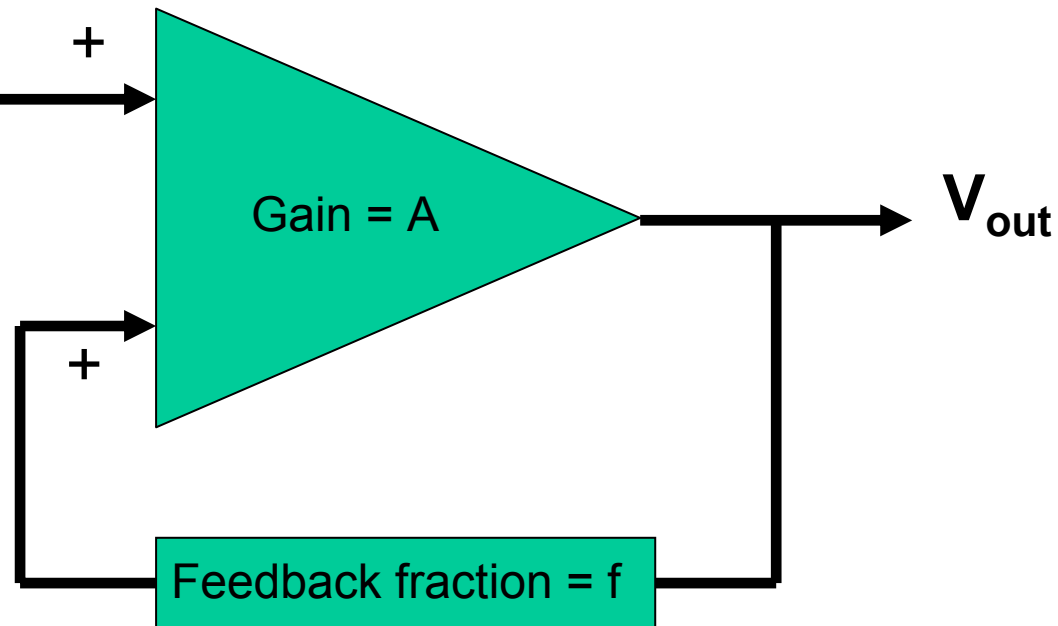
$$V_{out} = A V_{in} + (Af) A V_{in} + (Af)^2 A V_{in} + (Af)^3 A V_{in} + (Af)^4 A V_{in} \dots$$

$$V_{out} = A V_{in} (1 + Af + (Af)^2 + (Af)^3 + (Af)^4 + \dots)$$

$$V_{out} = A V_{in} (1 - Af)^{-1}$$

$$V_{out} (1 - Af) = A V_{in}$$

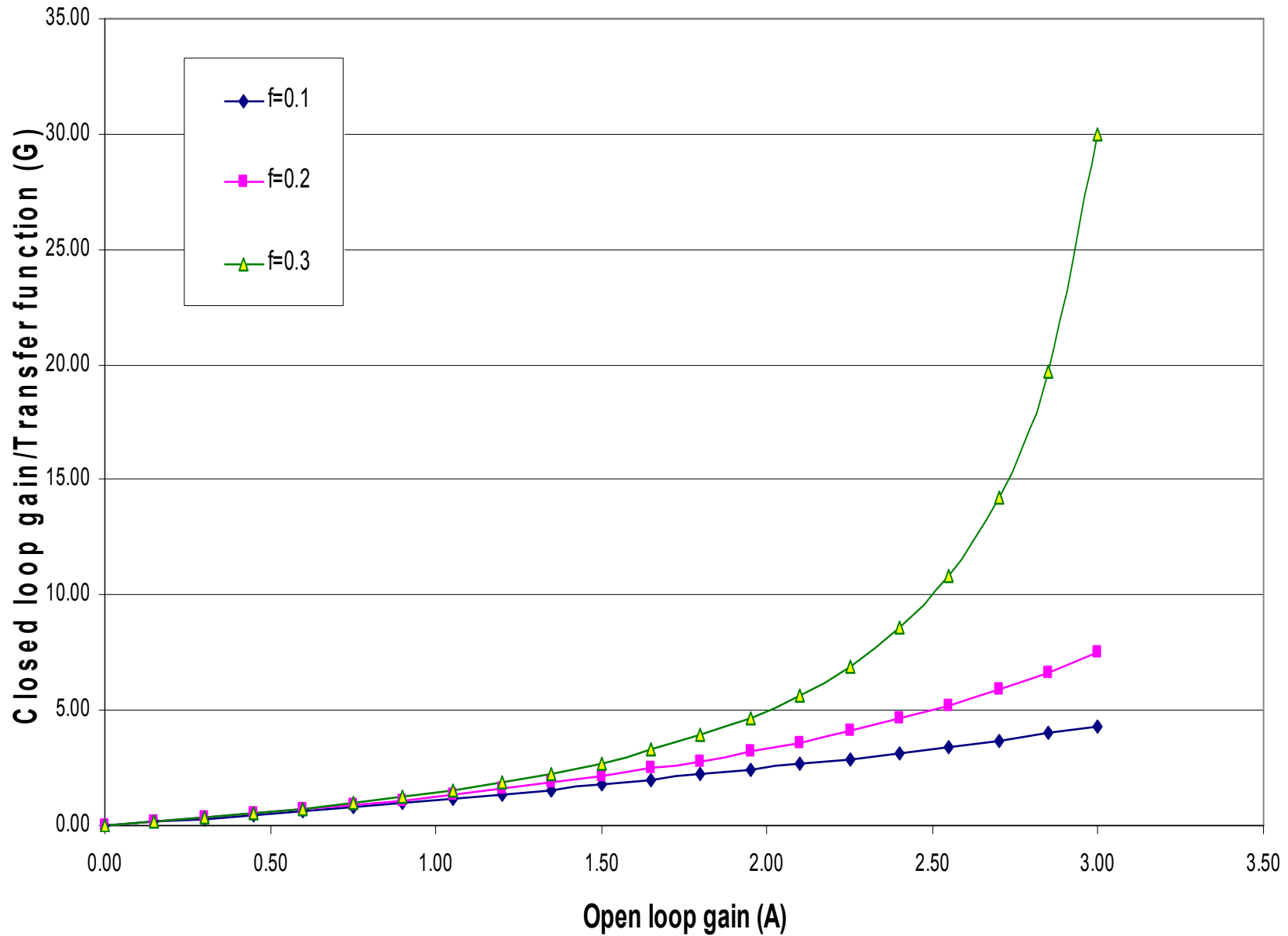
$$V_{out} - Af V_{out} = A V_{in}$$



$$V_{out} = A (V_{in} + f V_{out}) \quad \therefore G = V_{out}/V_{in} = A / (1 - Af) = A / (1 - B)$$

so $G > A$, and $G \rightarrow \infty$ as $Af \rightarrow 1$ i.e. system becomes *unstable*

Positive Feedback: dependence on open loop gain



Copyright statement

- ◆ This resource was created by the University of Southampton and released as an open educational resource through the 'C-change in GEES' project exploring the open licensing of climate change and sustainability resources in the Geography, Earth and Environmental Sciences. The C-change in GEES project was funded by HEFCE as part of the JISC/HE Academy UKOER programme and coordinated by the GEES Subject Centre.
- ◆ This resource is licensed under the terms of the **Attribution-Non-Commercial-Share Alike 2.0 UK: England & Wales** license (<http://creativecommons.org/licenses/by-nc-sa/2.0/uk/>).
- ◆ However the resource, where specified below, contains other 3rd party materials under their own licenses. The licenses and attributions are outlined below:
- ◆ The University of Southampton and the National Oceanography Centre, Southampton and its logos are registered trade marks of the University. The University reserves all rights to these items beyond their inclusion in these CC resources.
- ◆ The JISC logo, the C-change logo and the logo of the Higher Education Academy Subject Centre for the Geography, Earth and Environmental Sciences are licensed under the terms of the Creative Commons Attribution -non-commercial-No Derivative Works 2.0 UK England & Wales license. All reproductions must comply with the terms of that license.