Basic Linear System Dynamics

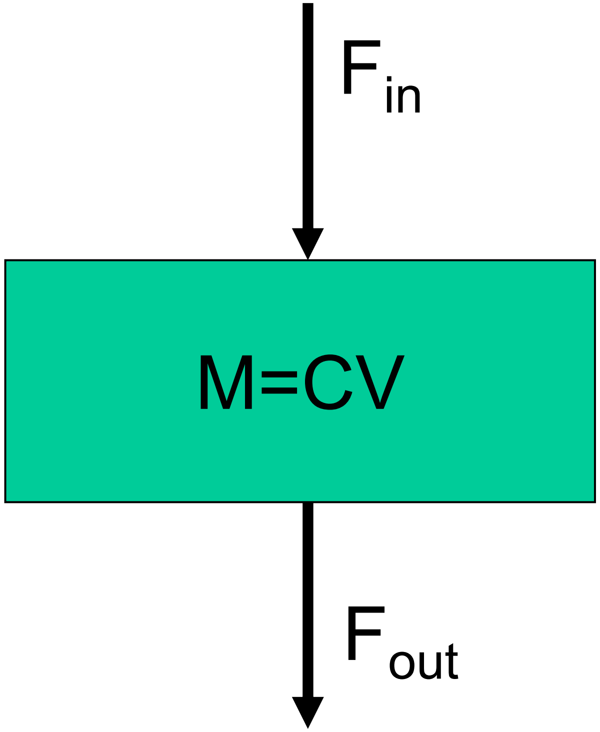
Reservoirs & Fluxes
Steady-states & Transients
Residence & Response times
System Gain & Frequency Response
Feedbacks, Loops & Oscillations

Originally by John Shepherd, Modified by Kevin Oliver, 2010

Basic System Dynamics

- Most theory is for ***linear*** systems
  - ***Non-linear*** systems are much trickier, but approximations to linear systems can be obtained for *small perturbations*.
- Concepts: Fluxes and inventories
- ***Steady-state*** (equilibrium) & ***transient*** (disequilibrium) conditions (*statics vs. dynamics*)
- Reservoir effects: finite response times
  - Residence times, Transit times, Mixing times, Reaction times, etc
- Multiple/continuous reservoirs (e.g. the ocean)
  - *Distributions* of response times & ages

Single reservoir
residence time & response time (1)

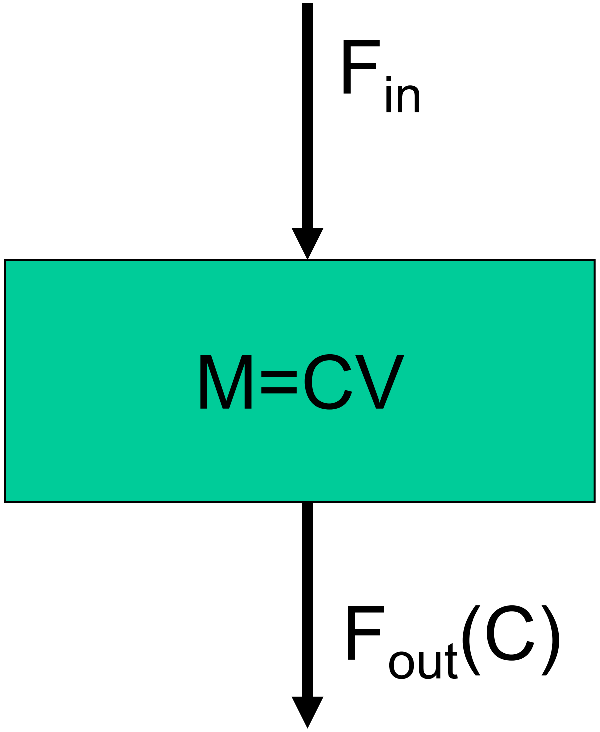


V = volume of reservoir (fixed)
C = concentration (quantity per unit volume)
M = total mass or quantity (= inventory) (= "standing stock")
M=CV

F=flux (quantity per unit time)
NB: may (often) be per unit area
Change = Input - Output
$\therefore\ dM/dt = F_{in} - F_{out}$
at ***steady-state*** $dM/dt = 0$
$\therefore\ F_{in} = F_{out}$ and $\tau_{res} = M/F_{in} = M/F_{out}$

NB: Quantity may be measured as mass, moles, etc, etc…
One ***must*** ensure dimensional consistency (i.e. the ***units*** check out OK)

Single reservoir residence time & response time (2)



In general F_{in} and F_{out} may depend on C
e.g. in a ***first-order removal rate*** process

$F_{out} = k'C = kCV$
where $k' = kV$, where $[k] = 1/[\text{time}]$

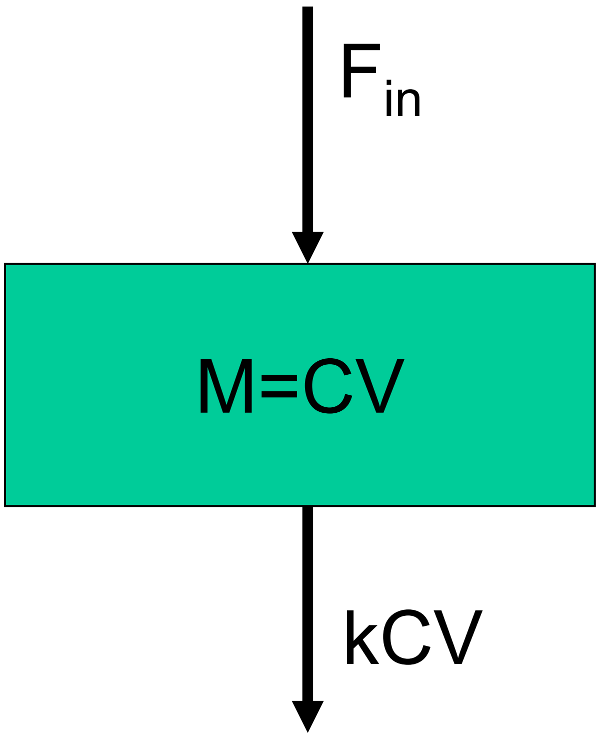
$dM/dt = d(CV)/dt = F_{in} - kCV$

$dC/dt = S - kC$
where ***source term*** $S = (F_{in}/V)$
(quantity per unit volume per unit time)

Solution is ***exponential approach to steady-state***
$C = (S/k)\ \{1 - \exp(-kt)\}$
Response time = residence time = $\tau_{res} = 1/k$

NB: (1) for radio-active decay $k = \lambda$ (the radio-active decay constant)
(2) for multiple (***parallel***) removal processes $k_{tot} = k_1 + k_2 + k_3 \ldots$
and hence $\tau_{res} = 1 / \{1/\tau_1 + 1/\tau_2 + 1/\tau_3\}$ like "resistances in parallel"
in this case the ***fastest*** process (with the smallest τ value) dominates

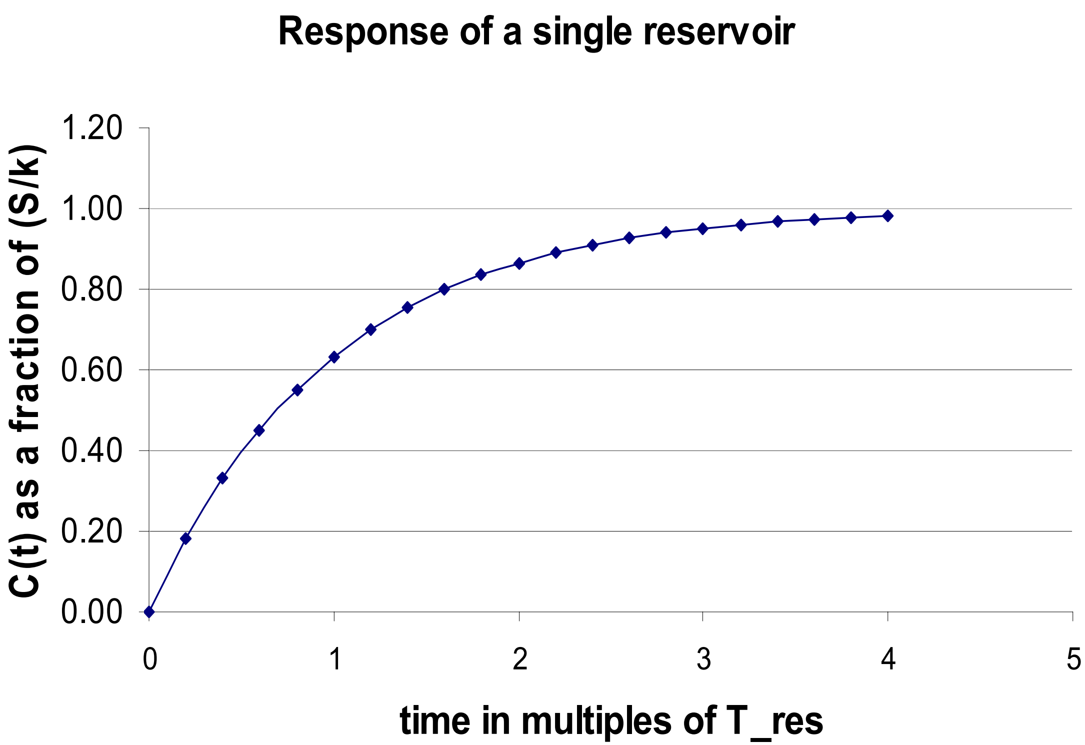
Single reservoir
residence time & response time (3)



Solution is ***exponential approach to steady-state***

$$C = (S/k)\ \{1 - \exp(-kt)\}$$

Response time = residence time = τ_{res} = 1/k



Characteristic system timescales

- **Residence time**; $\tau_{res} = 1/k\ (= 1/\lambda)$
- **Transit time** (advection); $\tau_{adv} = L/u$
  - Where L = length scale, u = velocity
- **Mixing time** (diffusion); $\tau_{mix} = L^2/2K$
  - Where K = diffusion coefficient
  - Usually due to turbulence
  - May differ (a lot) in different spatial dimensions
  - Horizontal mixing is a lot faster than vertical mixing
  - K values are ~ 10^8 times larger in horizontal than in vertical
- **Scavenging/deposition time**; $\tau_{dep} = D/v_{dep}$
  - Where D = depth scale, v_{dep} = deposition velocity
  - Also applies to **gas exchange**; so $\tau_g = D/v_g$
- **NB: Flux (per unit area) = velocity × concentration**
  - (with concentration expressed as quantity ***per unit volume***)
- ***With processes in series, the slowest process dominates***

Multiple reservoirs
"Mass" fluxes and exchanges

- Consider simple exchanges between reservoirs
- Exchange time τ_{exch} = Vol/Q
  - Where Q = u_{exch} × Area
  - Dimensions of Q are [Volume/Time]
  - i.e "mass" or volume flux, e.g. m^3/sec or Sv
  - Q and u_{exch} may be due to advective or diffusive exchange (or both)
- For multiple reservoirs (e.g. in a box model)
  - For the i^{th} box, with volume V_i , and exchange coefficients Q_{ij} between the i^{th} and the j^{th} boxes

$$\tau_i = V_i / \sum_j Q_{ij}$$

So the exchanges operate in parallel, and the fastest dominates

Basic System Dynamics

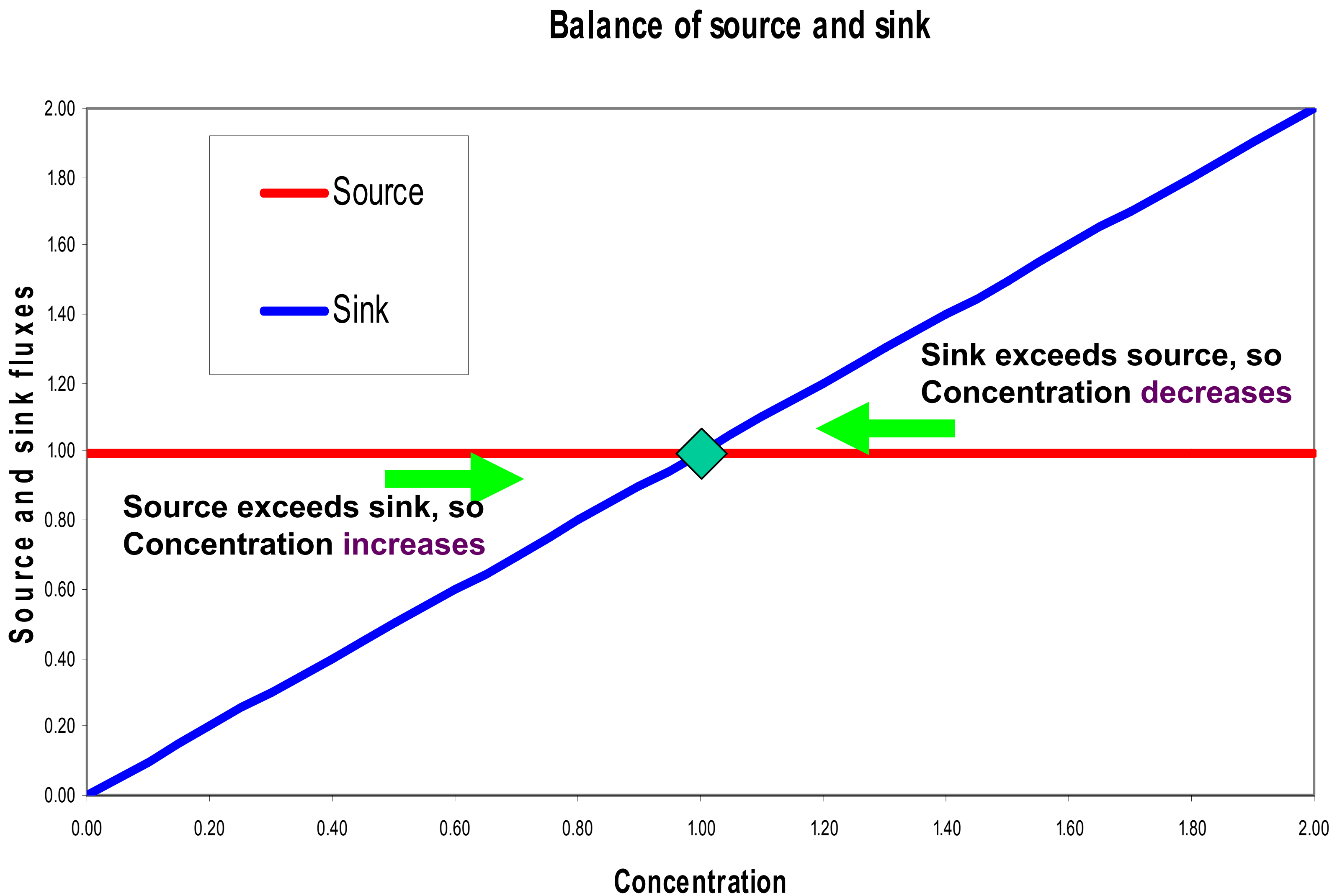
Feedbacks, Loops & Oscillations

Linear System Dynamics: Feedbacks

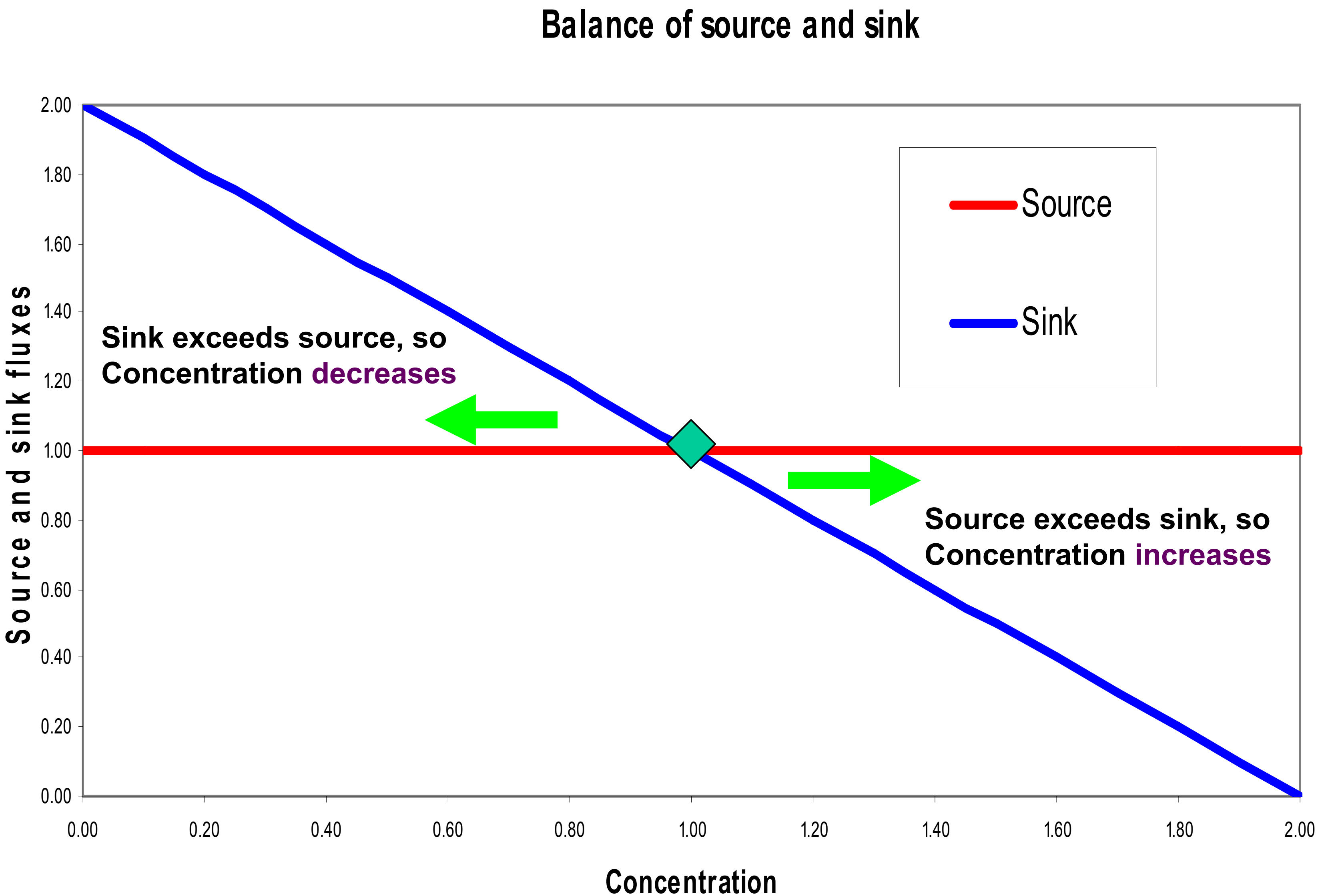
- ***Positive*** feedbacks – if these outweigh negative feedbacks:
  - multiple alternative states (separated by **repellors**)
  - hysteresis & rapid transitions (switching)

- ***Negative*** feedbacks – if these outweigh positive feedbacks:
  - if ***instantaneous*** : stabilisation, stable states (**attractors**)
  - if ***delayed*** : may get resonance (amplification) & possibly oscillations (at characteristic frequencies)

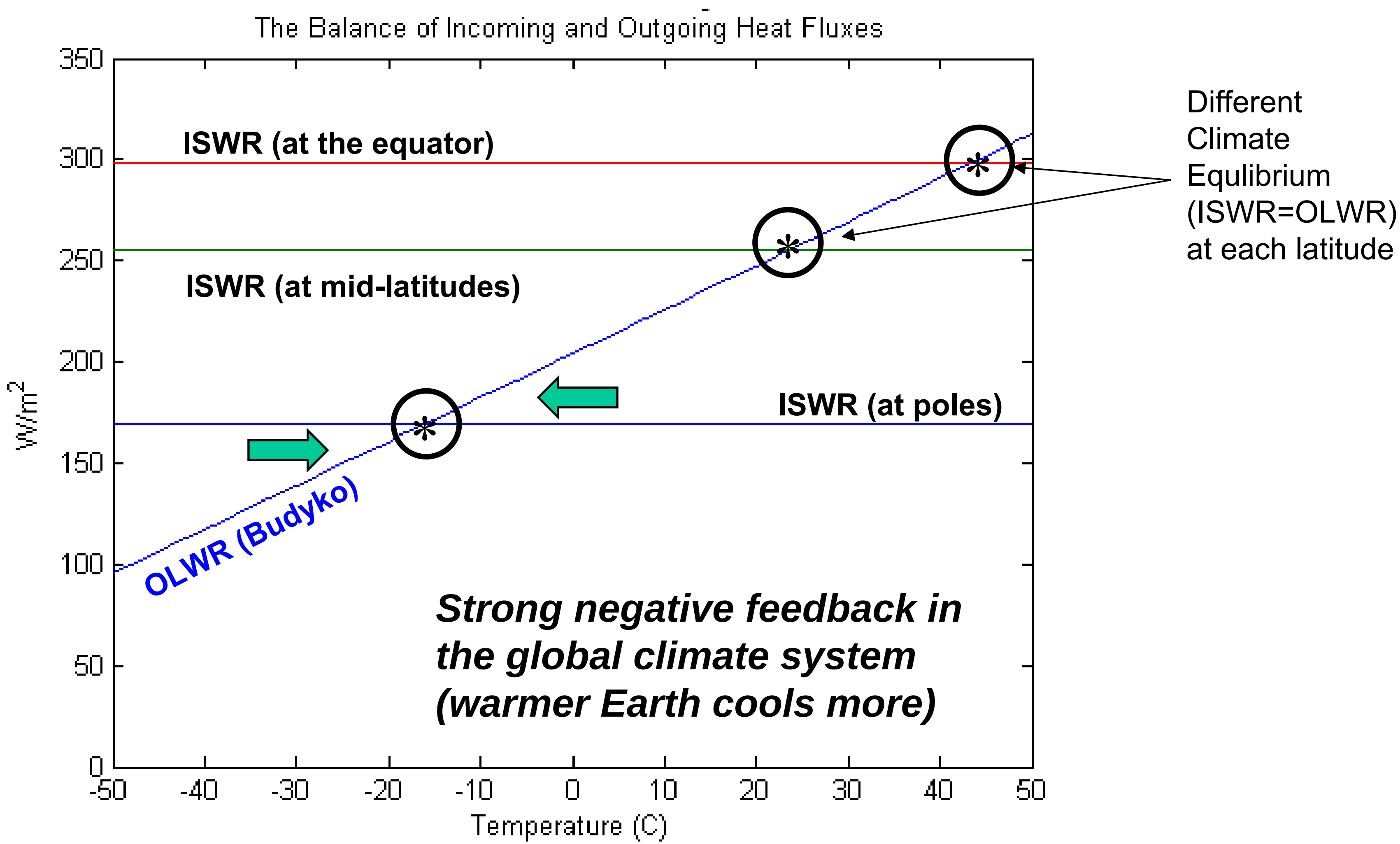
Negative feedback: constant source, but sink flux ***increases*** with concentration



Positive feedback: constant source, but sink flux ***decreases*** with concentration



Equating ISWR and OLWR (see Lecture 7)



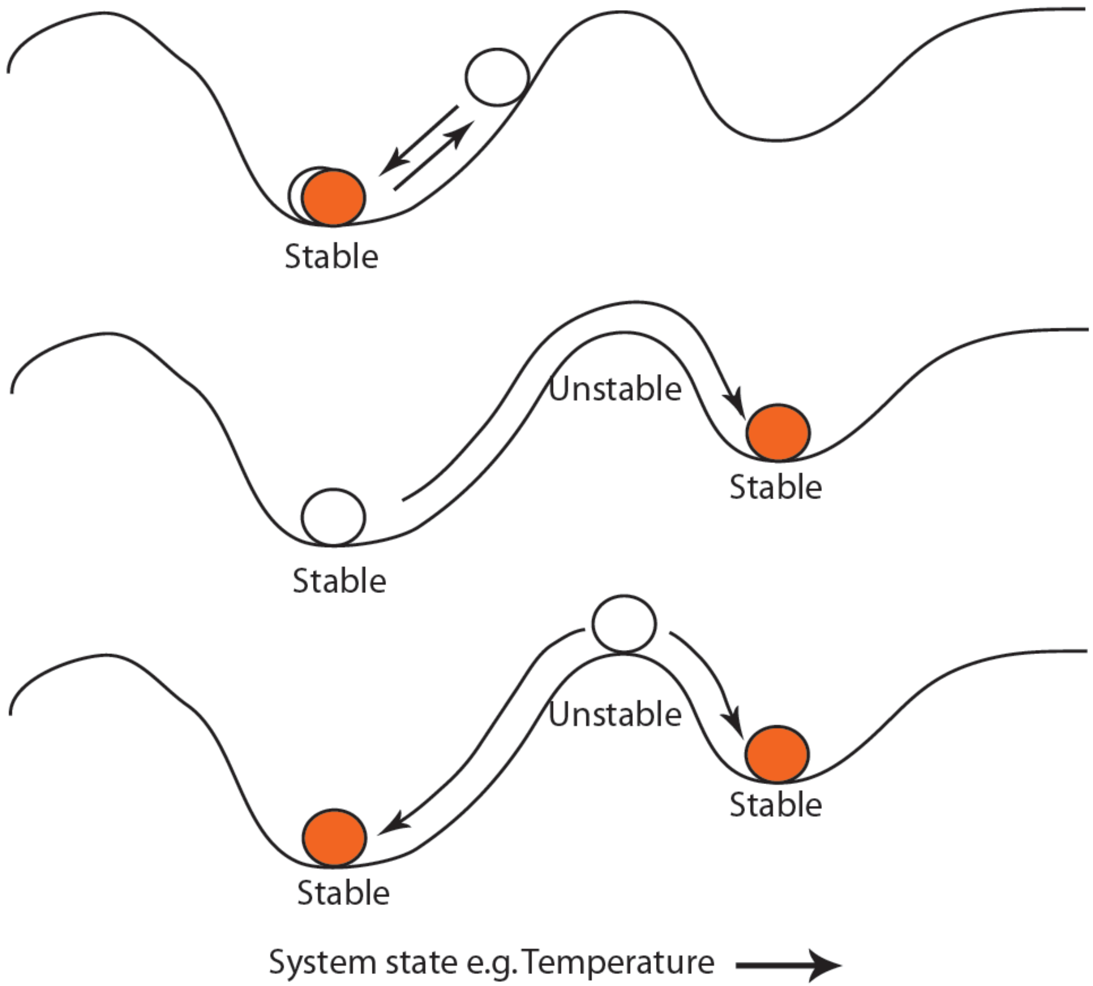
- If T < T*, ISWR > OLWR => warmer
- If T > T*, ISWR < OLWR => cooler

Feedbacks & response to perturbations

- If the concentration is perturbed from equilibrium…
- With ***negative*** feedback, the sink term changes so as to reduce the perturbation
  - And so restore the equilibrium
  - ∴ the feedback is ***stabilising***
  - ***The system "seeks" a stable steady-state***
- With ***positive*** feedback, the sink term changes so as to increase the perturbation
  - And so increase the dis-equilibrium
  - ∴ the feedback is ***destabilising***
  - ***If positive feedbacks outweigh negative feedbacks, the system perturbation grows irreversibly*** (though in practice the linearity of the system usually breaks down instead)

Stable, and unstable equilibria

(strictly, ***stationary points)*** of the system



Negative Feedback

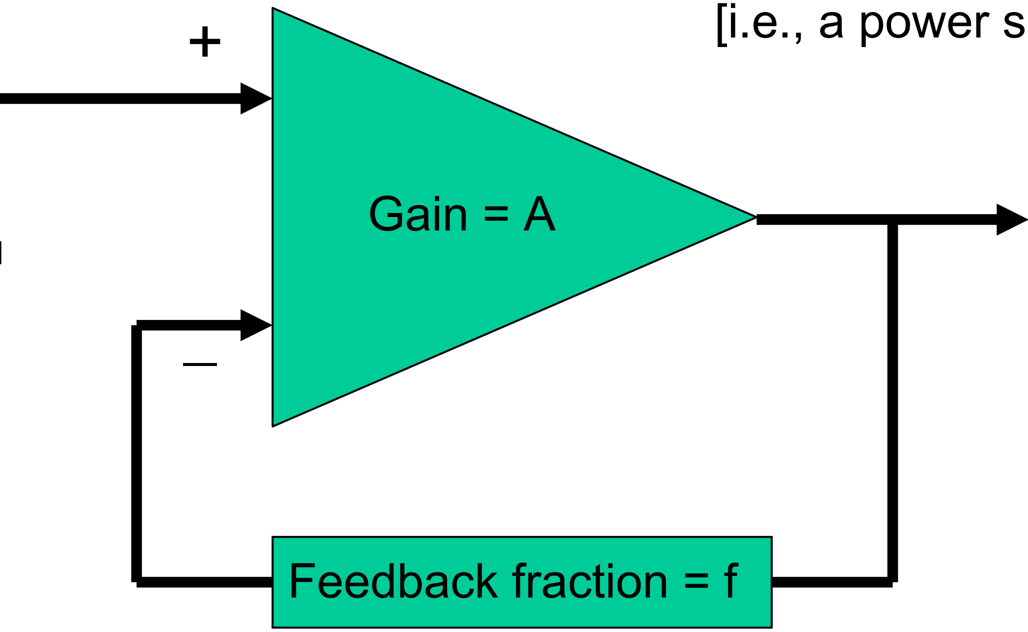
A = Open loop gain
f = feedback fraction
B = Af = Loop gain
G = Overall (closed loop) gain
= Transfer function

V_{out} = A V_{in} − (Af) A V_{in} + $(Af)^2$ A V_{in} − $(Af)^3$ A V_{in} + $(Af)^4$ A V_{in} …

V_{out} = A V_{in} (1 - Af + $(Af)^2$ - $(Af)^3$ + $(Af)^4$ - …)

[i.e., a power series]

V_{in}



V_{out}

V_{out} = A V_{in} $(1 + Af)^{-1}$

V_{out} (1 + Af) = A V_{in}

V_{out} + Af V_{out} = A V_{in}

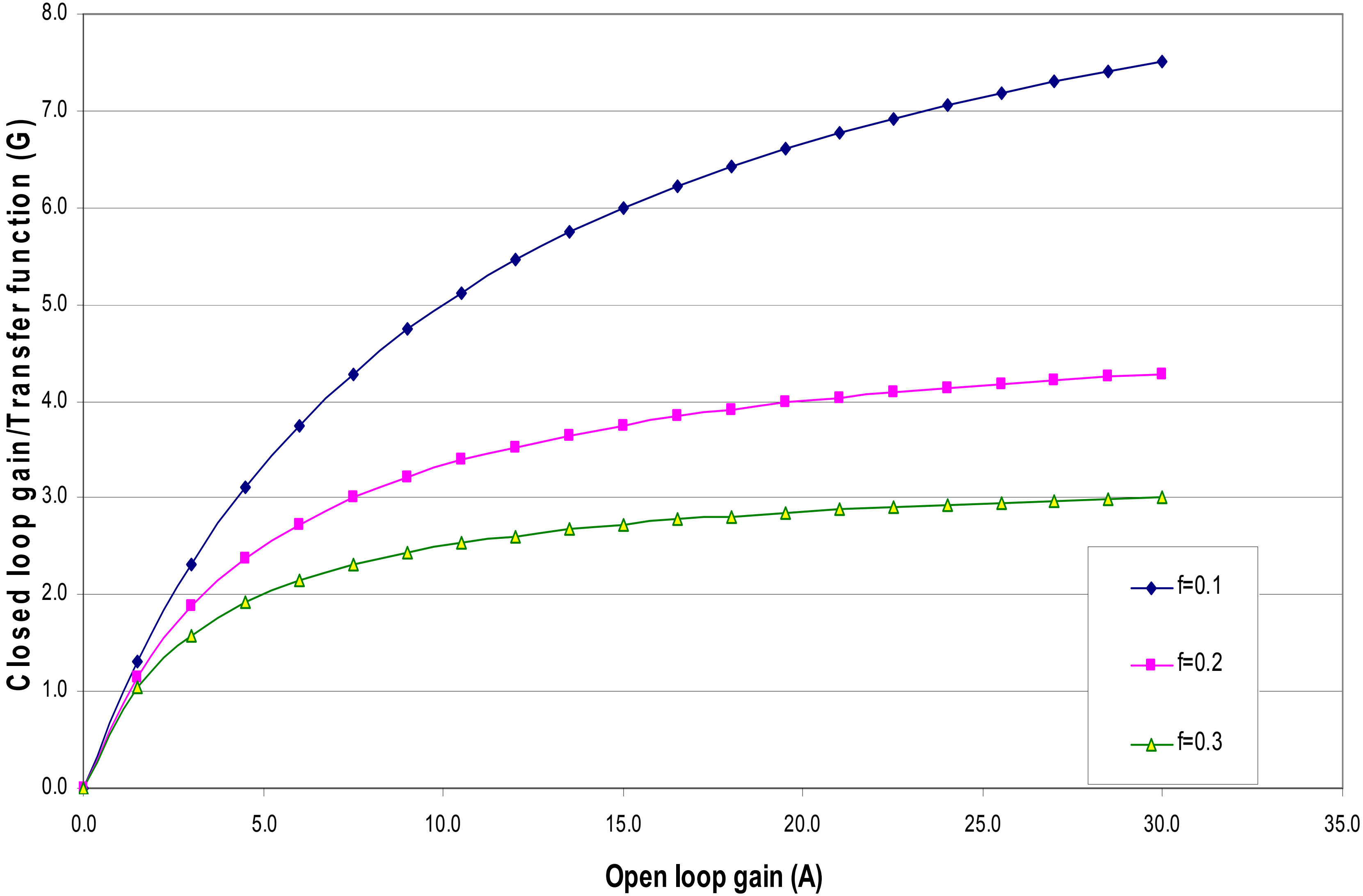
V_{out} = A (V_{in} – f V_{out})

∴ G → 1/f as A → ∞

∴ G = V_{out}/V_{in} = A / (1+Af) = A/(1+B)

so G << A, and is “stable” (independent of A)

Negative Feedback: dependence on open loop gain



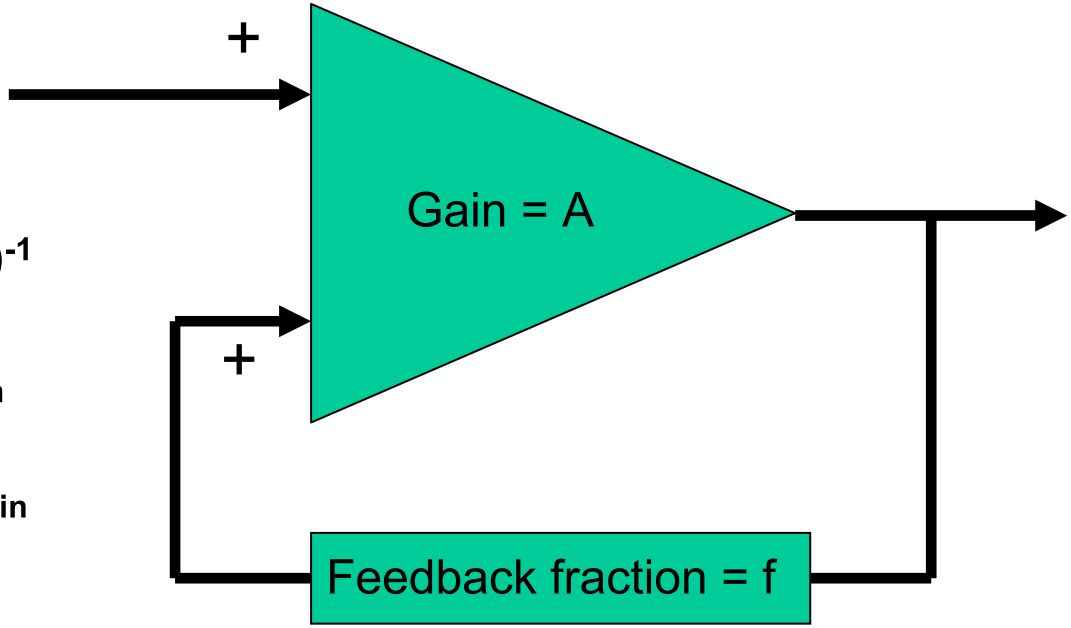
Positive Feedback

A = Open loop gain
f = feedback fraction
B = Af = Loop gain
G = Overall (closed loop) gain
= Transfer function

$$V_{out} = A\,V_{in} + (Af)\,A\,V_{in} + (Af)^2\,A\,V_{in} + (Af)^3\,A\,V_{in} + (Af)^4\,A\,V_{in} \ldots$$

$$V_{out} = A\,V_{in}\,(1 + Af + (Af)^2 + (Af)^3 + (Af)^4 + \ldots)$$

V_{in}



V_{out}

$$V_{out} = A\,V_{in}\,(1 - Af)^{-1}$$

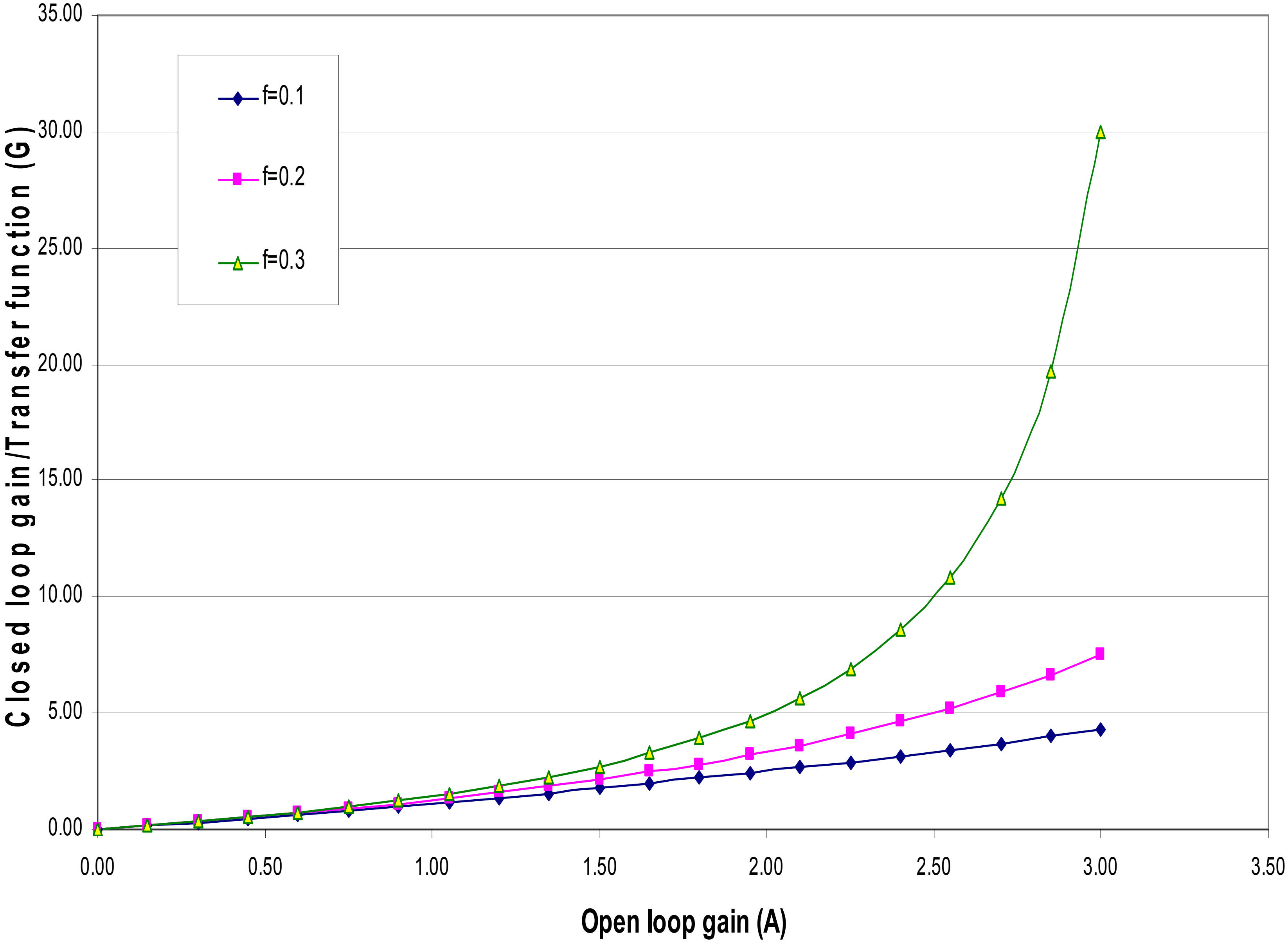
$$V_{out}\,(1 - Af) = A\,V_{in}$$

$$V_{out} - Af\,V_{out} = A\,V_{in}$$

$V_{out} = A\,(V_{in} + f\,V_{out})$ $\therefore\ G = V_{out}/V_{in} = A\,/\,(1-Af) = A/(1-B)$

so G > A, and G $\rightarrow \infty$ as Af $\rightarrow$ 1 i.e. system becomes *unstable*

Positive Feedback: dependence on open loop gain



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