Question

For each of the following functions described below, determine whether there is a solution to the given equation in the specified set.

1. $g'(a) = 0 = g'(b)$, where $a < b$ are real numbers and $g(x) = x^3 - 12\pi x^2 + 44\pi^2 x - 48\pi^3 + \cos(x) - 1$;

2. $f'(a) = 0$, where $f(x) = x^4 - \pi^3 x - \sin(x)$ and $a \in \mathbf{R}$;

3. $g'(x) = 0$ for at least $k-1$ distinct real numbers, where $g(x)$ is a differentiable function on $\mathbf{R}$ which vanishes at at least k distinct real numbers.

4. $x^3 + px + q = 0$ has exactly one real root for $p > 0$;

Answer

(In these problems, I've stopped explicitly checking the continuity and diffentiability hypotheses of the intermediate value property and of Rolle's theorem and the mean value theorem, because they have been checked so many times already and since they hold true for all the functions in this exercise.)

1. using the general mantra that two solutions to $g(x) = 0$ yield one solution to $g'(x) = 0$ via Rolle's theorem, let's see if we can find three solutions to $g(x) = 0$ for $g(x) = x^3 - 12\pi x^2 + 44\pi^2 x - 48\pi^3 + \cos(x) - 1$. Factoring, we see that $g(x) = (x - 2\pi)(x - 4\pi)(x - 6\pi) + \cos(x) - 1$, and so $g(2\pi) = g(4\pi) = g(6\pi) = 0$. By Rolle's theorem, there then exists a in $(2\pi, 4\pi)$ and b in $(4\pi, 6\pi)$ so that $g'(a) = g'(b) = 0$, as desired. (Also, note that the mixture of polynomial and trigonometric functions makes it unlikely that we would find solutions to $g'(x) = 0$ by direct calculation.)

2. a still slightly different method: calculating, we see that $f'(x) = 4x^3 - \pi^3 - \cos(x)$, and that $f'(-10) = -4000 - \pi^3 - \cos(-1000) < 0$ and that $f'(10) = 4000 - \pi^3 - \cos(1000) > 0$. Since f is continuous on $\mathbf{R}$, it is certainly continuous on the interval $[-10, 10]$, and so by the intermediate value property, there is some a in $(-10, 10)$ at which $f'(a) = 0$.

3. label the points at which g vanishes as $a_1 < a_2 < \cdots < a_n$. For each consecutive pair a_k, a_{k+1}, Rolle's theorem yields that there exists a point b_k between a_k and a_{k+1} at which $g'(b_k) = 0$. This yields $k-1$ points $b_1, \ldots, b_{k-1}$ at which the derivative $g'(x)$ vanishes, as desired.

4. let $h(x) = x^3 + px + q$. Suppose that h has two real roots; by Rolle's theorem, there is then a number c between these roots at which $h'(c) = 0$. However, calculating directly we see that $h'(x) = 3x^2 + p \geq p > 0$ for all $x \in \mathbf{R}$, and so there are no solutions to $h'(x) = 0$. Hence, there can be at most one root of h.

   To see that there is a root, we note that since h has odd degree (and since the coefficient of the highest degree term is positive), we have that $\lim_{x\to\infty} h(x) = \infty$ and $\lim_{x\to-\infty} h(x) = -\infty$. Hence, we can find a point a at which $h(a) > 0$ and a point b at which $h(b) < 0$, and the intermediate value property then implies that there is a point between a and b at which $h(x) = 0$.