Question

Find the vector equation and standard equation of the following planes:

(i) The plane Π_1 which passes through the point $P = (5, 7, -6)$ and has the normal vector $\mathbf{n} = 2\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}$;

(ii) The plane Π_2 which passes through the three points $P_1 = (4, 4, -2)$, $P_2 = (1, 0, -4)$, $P_3 = (7, 15, 2)$;

(iii) The plane Π_3 which passes through the point $P = (5, 0, -8)$ and is parallel to the plane with equation $4x + 4y - 14z = 11$.

Are any of the planes Π_1, Π_2, Π_3 parallel? Are any of them equal?

Answer

(i) Vector equation of the plane is $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$.

Where

$$\begin{aligned} \mathbf{n} &= \begin{pmatrix} 2 \\ 2 \\ -7 \end{pmatrix} \text{ is a normal vector to the plane.} \\ \mathbf{r} &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ is the possition vector of a general point in the plane.} \\ \mathbf{r_0} &= \begin{pmatrix} 5 \\ 7 \\ -6 \end{pmatrix} \text{ is the possition vector of a known point in the plane.} \end{aligned}$$

Hence $\begin{pmatrix} 2 \\ 2 \\ -7 \end{pmatrix} \cdot \left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 5 \\ 7 \\ -6 \end{pmatrix} \right] = 0$ or $\begin{pmatrix} 2 \\ 2 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} x-5 \\ y-7 \\ z+6 \end{pmatrix} = 0$

This gives the equation

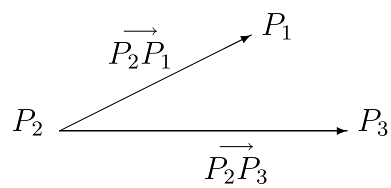
$$\begin{aligned} 2(x-5) + 2(y-7) - 7(z+6) &= 0 \\ 2x + 2y - 7z &= 10 + 14 + 42 = 66 \\ \text{so} \quad 2x + 2y - 7z &= 66. \end{aligned}$$

(ii) Plane passes through the point $P_2 = (1, 0, -4)$ with position vector $\mathbf{r_0} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$

Two vectors in the plane are:

$$\overrightarrow{P_2P_1} = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$$

$$\overrightarrow{P_2P_3} = \begin{pmatrix} 7 \\ 15 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ 15 \\ 6 \end{pmatrix}$$



A normal vector to the plane is

$$\mathbf{n} = \overrightarrow{P_2P_1} \times \overrightarrow{P_2P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 2 \\ 6 & 15 & 6 \end{vmatrix} = \begin{pmatrix} -6 \\ -6 \\ 21 \end{pmatrix}$$

Vector equation of the plane is: $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$.

$$\begin{pmatrix} -6 \\ -6 \\ 21 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y \\ z+4 \end{pmatrix} = 0$$

Standard equation of plane:

$$\begin{aligned} -6(x-1) - 6y + 21(z+4) &= 0 \\ -6x - 6y + 21z &= -6 - 84 = -90 \\ \text{so} \quad 2x + 2y - 7z &= 30. \end{aligned}$$

(iii) The plane π_3 passes through the point with position vector $\mathbf{r_0} = \begin{pmatrix} 5 \\ 0 \\ -8 \end{pmatrix}$ A normal to the plane is $\mathbf{n} = \begin{pmatrix} 4 \\ 4 \\ -14 \end{pmatrix}$ (i.e. the same normal vector as its parallel plane $4x + 4y - 14z = 11$)

Vector equation of π_3: $\begin{pmatrix} 4 \\ 4 \\ -14 \end{pmatrix} \cdot \begin{pmatrix} x-5 \\ y \\ z+8 \end{pmatrix} = 0$ Standard equation of π_3:

$$\begin{aligned} 4(x-5) + 4y - 14(z+8) &= 0 \\ 4x + 4y - 14z &= 20 + 112 = 132 \\ \text{so} \quad 2x + 2y - 7z &= 66. \end{aligned}$$

The planes π_1, π_2, π_3 are all parallel, because their normal vectors are all parallel to $\mathbf{n} = \begin{pmatrix} 2 \\ 2 \\ -7 \end{pmatrix}$ The planes π_1 and π_3 are equal because the are parallel, and their equations are consistent.