Question

Suppose that f is continuous and that the sequence c, $f(c)$, $f(f(c))$, $f(f(f(c)))$, $\ldots$ converges to a. Prove that $f(a) = a$.

Answer

first, for the sake of notational clarity, define the n-fold composition of f with itself by $f^{\circ n}$, so that $f^{\circ n} = f \circ f^{\circ(n-1)}$. The hypothesis can then be restated as saying that the sequence $\{f^{\circ n}(c)\}$ converges to a. Now, apply f to both sides. Since f is continuous, the sequence $\{f(f^{\circ n}(c))\}$ converges to $f(a)$, by the note below. However, since $f(f^{\circ n}(c)) = f \circ f^{\circ n}(c) = f^{\circ(n+1)}(c)$, the sequence $\{f(f^{\circ n}(c))\}$ is the same as the sequence $\{f^{\circ n}(c)\}$ with the first term removed, and so $\{f(f^{\circ n}(c))\}$ converges to a as well. Hence, since $\{f(f^{\circ n}(c))\}$ converges to both a and $f(a)$, we have that $a = f(a)$.

Note:

Suppose that f is continuous and that the sequence $\{a_n\}$ converges to a. Then, the sequence $\{f(a_n)\}$ converges to $f(a)$.