FOURIER SERIES

We consider trigonometrical series which have period 2π. Consider the series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

and suppose it converges uniformly with respect to x in $[-\pi, \pi]$ with sum $f(x)$, which will be continuous in $[-\pi\pi]$, and periodic - 2π. Then

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\, dx &= \frac{1}{2}a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n \cos nx + b_n \sin nx)\, dx = \pi a_0 \\
\int_{-\pi}^{\pi} f(x) \cos mx\, dx &= \frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos mx\, dx \\
&\quad + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n \cos nx + a_n \sin nx) \cos mx\, dx \\
&= a_m \int_{-\pi}^{\pi} \cos^2 mx\, dx = \pi a_m
\end{aligned}$$

Similarly $\int_{-\pi}^{\pi} f(x) \sin mx\, dx = \pi b_m$
Hence

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(f) \cos mx\, dx \quad m = 0, 1, \ldots \tag{1}$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx\, dx \quad m = 1, 2, \ldots \tag{2}$$

We can write the coefficients in the form

$$a_n + ib_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\, dx \quad n = 1, 2, \ldots$$

Now suppose $f(x) \in \mathcal{L}(-\pi\pi)$ and is periodic - 2π define constants $a_n,\ b_n$ by the relations (1), (2) above (Euler, Fourier formulae)
This gives rise to the series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \sim f(x)$$

called the Fourier Series of $f(x)$.
If f is an even function $b_m = 0$ $a_m = \frac{2}{\pi}\int_0^{\pi} f(x) \cos mx\, dx$
If f is an odd function $a_m = 0$ $b_m = \frac{2}{\pi}\int_0^{\pi} f(x) \sin mx\, dx$

Problem 1 Suppose f continuous and periodic and periodic -2π and $f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$.

Suppose the Fourier Series converges to $\phi(x)$ uniformly in $[-\pi\ \pi]$. Is it true the $f(x) = \phi(x)$?

By the definition of the coefficients, we get

$$\int_{-\pi}^{\pi} f(x)e^{inx}\,dx = \int_{-\pi}^{\pi} \phi(x)e^{inx}\,dx \quad n = 0, 1, 2, \ldots$$

where f and ϕ are continuous.

Does it follow that $f(x) = \phi(x)$ for all x?

Problem 2 Can 2 different continuous functions, periodic - 2π have the same Fourier Series.

Problem 3 Is the Fourier Series of a continuous periodic function convergent?

Theorem 1 Riemann Lebesgue If $f \in \mathcal{L}(a\ b)$ then $\int_a^b f(x)e^{i\lambda x}\,dx \to 0$ as $\lambda \to \infty$.

Proof We first prove the result for f continuous. Define, without loss of generality, $f(x) = f(b)\ x > b, f(x) = f(a) < a$ then f is continuous everywhere.

$$\begin{aligned} I &= \int_a^b f(x)e^{i\lambda x}\,dx \\ &= -\int_a^b f(x)e^{i\lambda\left(x+\frac{\pi}{\lambda}\right)}\,dx \\ &= -\int_{a+\frac{\pi}{\lambda}}^{b+\frac{\pi}{\lambda}} f(t-\frac{\pi}{\lambda})e^{i\lambda t}\,dt \\ &= -\int_a^b f\left(t-\frac{\pi}{\lambda}\right)e^{i\lambda t}\,dt + \int_a^{a+\frac{\pi}{\lambda}} f(t-\frac{\pi}{\lambda})e^{i\lambda t}\,dt \\ &\quad -\int_b^{b+\frac{\pi}{\lambda}} f\left(t-\frac{\pi}{\lambda}\right)e^{i\lambda t}\,dt \\ &= I_1 + I_2 - I_3 \end{aligned}$$

Therefore

$$2I = \int_a^b \left\{f(x) - f\left(x - \frac{\pi}{\lambda}\right)\right\} e^{i\lambda x}\,dx + I_2 - I_3$$

$$\left|\int_a^b \left\{f(x) - f\left(x - \frac{\pi}{\lambda}\right)\right\} e^{i\lambda x}\, dx\right| \leq \int_a^b \left|f(x) - f\left(x - \frac{\pi}{\lambda}\right)\right| dx$$

$\to 0$ as $\lambda \to \infty$ by uniform continuity of f.

$$|I_2| \leq \int_a^{a+\frac{\pi}{\lambda}} \left|f\left(t - \frac{\pi}{\lambda}\right)\right| dx < \frac{\pi}{\lambda} M \quad M = \sup_{a \leq x \leq b} f(x) < \infty$$

provided $\frac{\pi}{\lambda} < b - a$ therefore $I_2 \to 0$ as $\lambda \to \infty$.

Sum $I_3 \to 0$ as $\lambda \to \infty$

Therefore $I \to 0$ as $\lambda \to \infty$.

Now if $f \in \mathcal{L}(ab)$ and $\varepsilon > 0$ is given, there is a function $\phi(x) = \phi_\varepsilon(x)$, continuous in $[a\ b]$ such that $\int_a^b |f(x) - \phi(x)|\, dx < \varepsilon$. Therefore

$$\begin{aligned}
\int_a^b f(x) e^{i\lambda x}\, dx &= \int_a^b \phi(x) e^{i\lambda x} + \int_a^b \{f(x) - \phi(x)\} e^{i\lambda x}\, dx \\
&= I_1 + I_2
\end{aligned}$$

$|I_2| < \varepsilon$ for all λ $|I_1| < \varepsilon$ $\lambda > \lambda_0$ by the first part of the proof. Hence the result.

Corollary 1 If $a \leq a' \leq b' \leq b$ then $\int_{a'}^{b'} f(x) e^{i\lambda x}\, dx \to 0$ as $\lambda \to \infty$ uniformly in a' b'

Corollary 2

$$a_n + ib_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) e^{i\lambda x}\, dx \to 0 \text{ as } n \to \infty.$$

This result may be proved in a more elementary way if f is continuous as follows.

$$\begin{aligned}
\pi(a_n + ib_n) &= \int_{-\pi}^{\pi} f(x) e^{inx}\, dx \\
&= -\int_{-\pi}^{\pi} f(x) e^{in\left(x + \frac{\pi}{n}\right)}\, dx \\
&= -\int_{-n+\frac{\pi}{n}}^{n+\frac{\pi}{n}} f\left(t - \frac{\pi}{n}\right) e^{int}\, dt
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{-\pi}^{\pi} f\left(x - \frac{\pi}{n}\right) e^{n\lambda x} \\
&\quad \text{by periodicity of integrand.} \\
\text{Therefore } 2\pi(a_n + ib_n) &= \int_{-\pi}^{\pi} \left\{ f(x) - f\left(x - \frac{\pi}{n}\right) \right\} e^{inx}\, dx \\
\text{therefore } |2\pi(a_n + ib_n)| &\leq \int_{-\pi}^{\pi} \left| f(x) - f\left(x - \frac{\pi}{n}\right) \right| dx
\end{aligned}
$$

$\to 0$ as $n \to \infty$ by uniform continuity of the integrand.

Convergence of Fourier Series at $t = x$ Suppose $f(t) \in \mathcal{L}(-\pi\ \pi)$ and is periodic - 2π and suppose $f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$

$$
\begin{aligned}
S_n \equiv S_n(x) \equiv S_n(x; f) &= \frac{1}{2}a_0 + \sum_{\nu=1}^{n} a_\nu \cos \nu x + b_\nu \sin \nu x \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, dt + \sum_{\nu=1}^{n} \frac{\cos \nu x}{\pi} \int_{-\pi}^{\pi} f(t) \cos \nu t\, dt \\
&\quad + \sum_{\nu=1}^{n} \frac{\sin \nu x}{\pi} \int_{-\pi}^{\pi} f(t) \sin \nu t\, dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \left\{ \frac{1}{2} \sum_{\nu=1}^{n} (\cos \nu x \cos \nu t + \sin \nu x \sin \nu t) \right\} dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \left\{ \frac{1}{2} \sum_{\nu=1}^{n} \cos \nu(t - x) \right\} dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \left\{ \frac{\sin\left(n + \frac{1}{2}\right)(t - x)}{2 \sin \frac{1}{2}(t - x)} \right\} dt \\
&\quad \text{(Dirichlet's Integral)} \\
&= \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(x + u) \frac{\sin\left(n + \frac{1}{2}\right)u}{2 \sin \frac{1}{2}u}\, du \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x + u) \frac{\sin\left(n + \frac{1}{2}\right)u}{2 \sin \frac{1}{2}u}\, du
\end{aligned}
$$

The sum $\frac{1}{2} + \sum_1^n \cos \nu u = \frac{\sin\left(n+\frac{1}{2}\right)u}{2 \sin \frac{1}{2}u} = D_n(u)$ is called Dirichlet's Kernel.

$$
\frac{1}{\pi}\int_{-\pi}^{\pi} f(x + u) D_n(u)\, du = \frac{1}{\pi}\int_{0}^{\pi} \left\{ f(x + u) + f(x - u) \right\} D_n(u)\, du
$$

We have at $t = x$

$$
\begin{aligned}
f(t) = f(x+u) &= \frac{1}{2}\{f(x+u) + f(x-u)\} + \frac{1}{2}\{f(x+u) - f(x-u)\} \\
&= \phi_x(u) + \psi_x(u)
\end{aligned}
$$

ϕ_x =even part of $f(t)$ with respect to $t = x$

ψ_x =odd part of $f(t)$ with respect to $y = x$.

Dirichlet's integral becomes $\frac{2}{\pi}\int_0^\pi \phi_x(t) D_n(t)\,dt$.

Theorem 2 The convergence of a Fourier series is a 'local' property of the function when $f(t) \in \mathcal{L}(-\pi, \pi)$ and is periodic - 2π.

Proof Let $0 < \delta < \pi$

$$
\begin{aligned}
\pi S_n(x) &= \int_{-\pi}^{\pi} f(x+t) D_n(t)\,dt \\
&= \int_{-\pi}^{-\delta} + \int_{-\delta}^{\delta} + |int_\delta^\pi = I_1 + I_2 + I_3 \\
I_3 &= \int_\delta^\pi \frac{f(x+u)}{2\sin\frac{1}{2}u} \sin(n + \frac{1}{2}) u\,du
\end{aligned}
$$

$\rightarrow 0$ by Riemann Lebesgue theorem as $n \rightarrow \infty$ since $\frac{f(x+u)}{2\sin\frac{1}{2}u} \in \mathcal{L}(\delta\pi)$

Similarly $I_1 \rightarrow 0$ as $n \rightarrow \infty$.

Hence the convergence of the Fourier series at $t = x$ depends only on the behaviour of f in an arbitrarily small interval about $t = x$.

Theorem 3 If $f \in \mathcal{L}(-\pi\ \pi)$ and is periodic - 2π, and $a < b$, then the uniform convergence of the Fourier Series in $[a\ b]$ depends only on the function in any interval $(a - \delta, a + \delta)$ $\delta > 0$.

Proof Suppose $a \leq x \leq b$ and $\delta > 0$

$$
\begin{aligned}
\pi S_n(x) &= \int_{-\pi}^{\pi} f(x+t) D_n(t)\,dt \\
&= \int_{-\pi+x}^{\pi+x} f(t) D_n(t-x)\,dt \\
&= \int_{\pi+x}^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{x+\pi} = I_1 + I_2 + I_3
\end{aligned}
$$

$$\begin{aligned} I_3 &= \int_{x+\delta}^{x+\pi} f(t)\frac{\sin\left(n+\frac{1}{2}\right)(t-x)}{2\sin\frac{1}{2}(t-x)}\,dt \\ &= \frac{1}{2\sin\frac{1}{2}\delta}\int_{x+\delta}^{\zeta} f(t)\sin\left(n+\frac{1}{2}\right)(t-x)\,dt \text{ 2nd MVT} \\ &= \frac{\cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{1}{2}\delta}\int_{x+\delta}^{\zeta} f(t)\sin\left(n+\frac{1}{2}\right)t\,dt \\ &\quad -\frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{1}{2}\delta}\int_{x+\delta}^{\zeta} f(t)\cos\left(n+\frac{1}{2}\right)t\,dt \end{aligned}$$

$a \le a+\delta \le x+\delta \le \zeta \le x+\pi \le b+\pi$
$\left|\cos\left(n+\frac{1}{2}\right)x\right| \le 1 \quad \left|\sin\left(n+\frac{1}{2}\right)x\right| \le 1$
Therefore $I_3 \to 0$ uniformly with respect to x as $n \to \infty$ by Riemann Lebesgue theorem. Similarly $I_1 \to 0$ uniformly with respect to x as $n \to \infty$.

hence the uniform convergence depends only on the behaviour of f in $(x-\delta, x+\delta)$ $a \le x \le b$ i.e. in $(a-\delta, b+\delta)$.

Note

$$S_n = \frac{2}{\pi}\int_0^{\pi} \phi_x(t)D_n(t)\,dt.$$

s - const is even therefore $S = \frac{2}{\pi}\int_o^{\pi} sD_n(t)\,dt$ therefore

$$S_n - S = \frac{2}{\pi}\int_0^{\pi}\{\phi_x(t) - S\}D_n(t)\,dt$$

Theorem 4 Dimi's Test If $f \in \mathcal{L}(-\pi\ \pi)$ and is periodic - 2π and if $\exists S | \frac{\phi_x(t)-S}{t} \in \mathcal{L}(0\ \delta)$ $\delta > 0$ then the Fourier series of f converges to S at $t = x$.

Proof

$$\begin{aligned} \frac{\pi}{2}(S_N(x) - S) &= \int_0^{\pi}(\phi_x(t) - S)D_n(t)\,dt \\ &= \int_0^{\pi}\frac{\phi_x(t)-S}{2\sin\frac{1}{2}t}\sin\left(n+\frac{1}{2}\right)t\,dt \\ \frac{\phi_x(t)-S}{2\sin\frac{1}{2}t} &= \frac{\phi_x(t)-S}{t}.\frac{t}{2\sin\frac{1}{2}t}. \end{aligned}$$

If $g(t) = \frac{t}{2\sin\frac{1}{2}t}$ $o < t \le \pi$, and $g(0) = 1$ then g is continuous in $[0\ \pi]$ Therefore $\frac{\phi_x(t)-S}{2\sin\frac{1}{2}t} \in \mathcal{L}(0, \delta)$ and also $\frac{\phi_x(t)-S}{2\sin\frac{1}{2}t} \in \mathcal{L}(\delta\ \pi)$ therefore $\frac{\phi_x(t)-S}{2\sin\frac{1}{2}t} \in \mathcal{L}(0,\ \pi)$.

So by Riemann-Lebesgue Theorem $\frac{\pi}{2}(S_n(x) - S) \to 0$ as $n \to \infty$ i.e. $S_n(x) \to S$ as $n \to \infty$.

Corollary I If $f(t) \in \mathcal{L}(-\pi\ \pi)$ and is periodic - 2π and if $f(t)$ is differentiable at $t = x$, then the Fourier series converges to $f(x)$ at $t = x$.

Proof

$$\frac{\phi_x(t) - f(x)}{t} = \frac{f(x+t) - f(x)}{2t} + \frac{f(x-t) - f(x)}{2t}$$

$\to \frac{1}{2}f'(x) - \frac{1}{2}f'(x) = 0$ as $t \to 0$ therefore $\exists\delta > 0 | \left|\frac{\phi_x(t)-f(x)}{t}\right| < 1\ 0 \leq t \leq \delta$ therefore $\frac{\phi_x(t)-f(x)}{t} \in \mathcal{L}(0, \delta)$.

Therefore by Dimi's test the Fourier series converges to $f(x)$.

Corollary II Lipschitz Condition $f(t) \in \mathcal{L}(-\pi\ \pi)$ and periodic - 2π and $f(x+t) - f(x) = O|t|^\alpha$ for some $\alpha \neq 0$ as $t \to 0$ then the Fourier series converges to $f(x)$ at $t = x$.

Proof

$$\begin{aligned}
|\phi_x(u) - f(x)| &\leq K|U|^\alpha \\
\text{therefore } \frac{|\phi_x(u) - f(x)|}{|U|} &\leq K|U|^\alpha \\
\text{therefore } \left|\int_0^\delta \frac{\phi_x(u) - f(x)}{u}\, du\right| &\leq K\int_0^\delta |U|^{\alpha-1}\, du = K\frac{\delta^\alpha}{\alpha} < \infty
\end{aligned}$$

Hence Dimi's condition is satisfied.

Modified forms of Dirichlet's integral if $f(t \in \mathcal{L}(-\pi\ \pi)$ and periodic - 2π, we have

$$S_n(x) - S = \frac{2}{\pi}\int_0^\pi \{\phi_x(t) - S\}\frac{\sin\left(n + \frac{1}{2}\right)t}{2\sin\frac{1}{2}t}\, dt$$

1. $\frac{1}{2\sin\frac{1}{2}t}$ may be replaced by $\frac{1}{t}$ with error O(1).

$$\begin{aligned}
h(t) &= \frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t} \\
&= \frac{t - 2\sin\frac{1}{2}t}{2t\sin\frac{1}{2}t}
\end{aligned}$$

$$= \quad \frac{t - 2\left(\frac{1}{2}t - \left(\frac{1}{2}\right)^3 \frac{t^3}{3!} + \ldots\right)}{2t\left(\frac{1}{2}t - \ldots\right)}$$

$$\sim \quad 2\frac{\left(\frac{1}{2}\right)^3}{3!}\frac{t^3}{t^2} = \frac{1}{24}t \text{ as } t \to 0$$

Therefore

$$\frac{2}{\pi}\int_0^\pi \{\phi_x(t) - S\}h(t)\sin\left(n + \frac{1}{2}\right)t\,dt$$

$\to 0$ as $n \to \infty$ by Riemann Lebesgue Theorem.

2. We may replace $\sin\left(n + \frac{1}{2}\right)t$ by $\sin nt$ with error $O(1)$.

$$\begin{aligned}
\sin\left(n + \frac{1}{2}\right)t - \sin nt &= 2\cos\left(n + \frac{1}{4}\right)t\sin\frac{1}{4}t \\
\text{therefore} \qquad & \frac{2}{\pi}\int_0^\pi \frac{\{\phi_x(t) - S\}}{t}\left\{\sin\left(n + \frac{1}{2}\right)t - \sin nt\right\}\,dt \\
&= \frac{2}{\pi}\int_0^\pi \{\phi_x(t) - S\}\frac{2\sin\frac{1}{4}t}{t}.\cos\left(n + \frac{1}{4}\right)t\,dt
\end{aligned}$$

$\to 0$ as $n \to \infty$ by Riemann-Lebesgue Theorem.

Functions of bounded variables Let $f(x)$ be defined on $[a, b]$.

Let $\Delta_N : a = x_0 < x_1 < \ldots < x_n = b$ and define $V_\Delta = \sum_{\nu=1}^n |f(x_\nu) - f(x_{\nu-1})|$

Let $\delta f(x_\nu) = f(x_\nu) - f(x_{\nu-1})$.

Define $\delta^+ f(x_\nu) = \frac{1}{2}\left[|\delta f(x_\nu)| + \delta(f x_\nu)\right] = \begin{matrix} \delta f(x_\nu) & \text{if} > 0 \\ 0 & \text{otherwise} \end{matrix}$

Define $\delta^+ f(x_\nu) = \frac{1}{2}\left[|\delta f(x_\nu)| - \delta(f x_\nu)\right] = \begin{matrix} -\delta f(x_\nu) & \text{if} > 0 \\ 0 & \text{otherwise} \end{matrix}$

Define $P_\Delta = \sum_{\nu=1}^n \delta^+ f(x_\nu),\ N_\Delta = \sum_{\nu=1}^n \delta^- f(x_\nu)$

Then $P_\Delta + N_\Delta = V_\Delta \quad P_\Delta - N_\Delta = f(b) - f(a)$

Define $P = P_a^b f(x) = \sup_\Delta P_\Delta$ to be the positive variation and $N = N_a^b f(x) = \sup_\Delta N_\Delta$ to be the negative variation.

$P_a^b + N_a^b = V_a^b \quad P_a^b - N_a^b = f(b) - f(a)$

Define $V(x) = V_a^x \quad P(x) = P_a^x \quad N(x) = N_a^x$

Theorem 5 If f is B.V. in $[a\ b]$ i.e. $V_a^b < \infty$ then $\exists$ functions f_1 and f_2 both increasing and bounded $|f(x) = f_1(x) - f_2(x)\ \ [a\ b]$

Proof $P(x) - N(x) = f(x) - f(a)$ therefore $f(x) = f(a) + P(x) - N(x)$.

$P(x)$ and $N(x)$ are increasing and bounded. Let $f_1(x) = f(a) + P(x)$ $f_2(x) = N(x)$.

Corollary A function of BV in $[a\ b]$ is continuous p.p. in $[a\ b]$.

Theorem 6 Jordan's Condition If $f(t) \in \mathcal{L}(-\pi\ \pi)$ and is periodic -2π and f is B.V in $[x-\delta, x+\delta]$ for some $\delta > 0$ then the Fourier series of f converges at $t = x$ to $\frac{1}{2}[f(x+) + f(x-)]$.

Proof $f(t)$ B.V. in $[x-\delta, x\delta] \Rightarrow \phi_x(t)$ B.V. in $[0\ \delta]$.

$$\begin{aligned} S_m(x) - \frac{1}{2}\{f(x+) + f(x-)\} &= \frac{2}{\pi}\int_0^\pi \{\phi_x(t) - \phi_x(0+)\} D_n(t)\,dt \\ &= \frac{2}{\pi}\int_0^\pi \{\phi_x(t) - \phi_x(0+)\}\frac{\sin nt}{t}\,dt + O(1) \end{aligned}$$

as $n \to \infty$.

By Theorem 5, $\phi_x(t) = \psi_x(t) = \theta_x(t)$, ψ, θ increasing and bounded so that $\psi_x(t) - \psi_x(0+)$ increasing and ≥ 0 in $[0\ \delta]$ and $\to 0$ as $t \to 0+$.

Therefore suppose without loss of generality that $\phi_x(t)$ increasing and bounded in $[0\ \delta]$ and $\phi_x(t) - \phi_x(0+) < \varepsilon$ for $0 \leq t \leq \eta < \delta$.

$$\begin{aligned} I &= \int_0^\pi \{\phi_x(t) - \phi_x(0+)\}\frac{\sin nt}{t}\,dt \\ &= \int_0^\eta + \int_\eta^\pi = I_1 + I_2 \\ |I_1| &= \left|\int_0^\eta \{\phi_x(t) - \phi_x(0+)\}\frac{\sin nt}{t}\,dt\right| \\ &= \{\phi_x(\eta) - \phi_x(0+)\}\left|\int_\zeta^\eta \frac{\sin ni}{t}\,dt\right| \text{ 2nd M.V.T} \\ &\leq \varepsilon \left|\int_{n\zeta}^{n\eta} \frac{\sin u}{u}\,du\right| \\ &\leq {}^2M\varepsilon \end{aligned}$$

for all $n \geq 0$ since $\int_0^x \frac{\sin u}{u}$ is continuous and $\to \frac{\pi}{2}$ as $x \to +\infty$ and is hence bounded by M. $I_2 \to 0$ as $n \to \infty$ by Riemann-Lebesgue theorem. Hence the result.

Examples 1.

$$f(t) = \frac{1}{\log\left|\frac{1}{t}\right|} \quad (0\ \delta] \quad f(0) = 0$$

satisfies Jordan's condition for $t \in [-\delta\ \delta]$ but doesn't satisfy Dimi's condition, since

$$\frac{\phi_0(t) - S}{t} = \frac{\left(\log\frac{1}{t}\right)^{-1} - S}{t} = \frac{1}{t\log\frac{1}{t}} - \frac{s}{t}$$

which cannot be integrated down to the origin.

2.

$$g(t) = |t|^\alpha \sin\left|\frac{1}{t}\right| \quad o < \alpha \leq 1$$

satisfies Dimi's condition

$$\frac{\phi_0(t) - 0}{t} = \left|\frac{t^\alpha \sin\frac{1}{t}}{t}\right| \leq |t|^{\alpha-1} \in \mathcal{L}(0\ \delta]$$

But not Jordan's condition.

$$\begin{aligned}
\phi_0(t) &= t^\alpha \sin\frac{1}{t} \\
\phi_0\left(\frac{1}{\left(2n+\frac{1}{2}\right)\pi}\right) &= \left[\frac{1}{\left(2n+\frac{1}{2}\right)\pi}\right]^\alpha \\
\phi_0\left(\frac{1}{\left(2n+\frac{1}{2}\right)\pi}\right) &= -\left[\frac{1}{\left(2n+\frac{1}{2}\right)\pi}\right]^\alpha \\
V_{\frac{1}{2N}}^{1}\phi_0(t) &\geq N_{\frac{1}{2N}}^{1} = \sum_1^{N-1}\left\{\frac{1}{\left(2n+\frac{1}{2}\right)\pi}\right\}^\alpha + \left\{\frac{1}{\left(2n-\frac{1}{2}\right)\pi}\right\}^\alpha \\
&\geq 2\pi^{-\alpha}\sum_1^{N-1}\frac{1}{(2n-1)\alpha}
\end{aligned}$$

$\to \infty$ as $N \to \infty$.

Theorem 7 If $f(t)$ is B.V. $[-\pi\ \pi]$ and periodic 2π then the Fourier series converges boundedly in $[-\pi, \pi]$.

Proof We prove $\exists M||s_n(x)| < M$ for all n and x

$S_n(x) \to \phi_x(0+)$ for all x by Jordan's test. We can suppose without loss of generality $f(t) > 0$ increasing and bounded in $(-\pi\ \pi)$

$$\begin{aligned}|S_n(x)| &= \left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(t-x)\right| = \frac{1}{\pi}f(\pi)\left|\int_{\zeta}^{\pi} D_n(t-x)\,dt\right| \\ &\leq f(\pi)(4\sqrt{\pi}+\frac{1}{2}(\pi-\zeta)) \leq f(\pi)(4\sqrt{\pi}+\frac{1}{2}\pi) = M\end{aligned}$$

Theorem 8 Of $f(t)$ is B.V in $[a\ b]$ $0 < b-a < 2\pi,\ f \in \mathcal{L}(-\pi\ \pi)$ periodic - 2π then the Fourier series converges boundedly in $[a+\eta, b-\eta]$ $0 < \eta < \frac{b-a}{2}$.

Proof $S_n(x) \to \phi_x(0+)$ $a < x < b$ by Jordan's test.

Suppose without loss of generality that $f(t)$ increasing and > 0 in $[a\ b]$.

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(t-x)\,dt$$

Choose η and suppose $x \in [a+\eta, b-\eta]$

$$\begin{aligned}S_n(x) &= \frac{1}{\pi}\int_a^{a+2\pi} f(t)D_n(t-x)\,dt \text{ by periodicity} \\ &= \frac{1}{\pi}\int_a^b + \frac{1}{\pi}\int_b^{a+2\pi} = I_1 + I_2 \\ |I_1| &= \frac{f(b)}{\pi}\left|\int_{\zeta}^{b} D_n(t-x)\,dt\right| \text{ 2nd M.V.T.} \\ &\leq \frac{f(b)}{\pi}\left[4\sqrt{\pi}+\frac{1}{2}\pi\right] = L \text{ for all } x, \eta \\ |I_2| &= \left|\frac{1}{\pi}\int_b^{a+2\pi} f(t)\frac{\sin\left(n+\frac{1}{2}\right)(t-x)}{2\sin\frac{1}{2}(t-x)}\,dt\right| \\ &\leq \frac{1}{\pi}\int_a^{a+2\pi} |f(t)|\frac{1}{|2\sin\frac{1}{2}(t-x)|}\,dt\end{aligned}$$

Now $\eta \leq t-x \leq 2\pi-\eta$ therefore $\frac{1}{2}\eta \leq \frac{1}{2}(t-x) \leq \pi - \frac{1}{2}\eta$ therefore

$$|I_2| \leq \frac{1}{2\pi}\frac{1}{2\sin\frac{1}{2}\eta}\int_b^{a+2n} |f(t)|\,dt \leq \frac{1}{\pi}\frac{1}{2\sin\frac{1}{2}\eta}\int_{-\pi}^{\pi} |f(t)|\,dt = C(\eta)$$

Hence the result.

Theorem 9 If $f \in \mathcal{L}(-\pi\ \pi)$ and is periodic - 2π, and if f B.V. in $[a\ b]$ $a < b$ then if $f(t)$ is continuous in $[a\ b]$ then the Fourier series converges uniformly to $f(t)$ in $[a+\eta, b-\eta]$.

Proof Suppose without loss of generality that $b-a < \pi$, and $f(t)$ increasing and > 0 in $[a\ b]$.

Let $0 < \eta < \frac{b-a}{2}$

Choose δ such that

(i) $0 < \delta < \eta$

(ii) $0 \leq f(t_2) - f(t_1) < \varepsilon$, whenever $\varepsilon > 0$ is given and $0 < t_2 - t_1 < \delta,\quad t_1, t_2 \in [a\ b]$

Suppose $x \in [a+\eta, b-\eta]$

$$\begin{aligned}
S_n(x) - f(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} D_n(t-x)[f(t)-f(x)]\,dt \\
&= \frac{1}{\pi}\int_{x-\pi}^{x+\pi} = \frac{1}{\pi}\left\{\int_{x-\pi}^{x-\delta} + \int_{x-\delta}^{x} + \int_{x}^{x+\delta} + \int_{x+\delta}^{x+\pi}\right\} \\
&= I_1 + I_2 + I_3 + I_4 \\
|I_3| &= \frac{1}{\pi}\left|\int_{x}^{x+\delta}\{f(t)-f(x)\}D_n(t-x)\,dt\right| \\
&= \frac{1}{\pi}[f(x+\delta)-f(x)]\left|\int_{\zeta}^{x+\delta} D_n(t-x)\,dt\right| < \varepsilon K_0 \\
|I_2| &= \frac{1}{\pi}\left|\int_{x-\delta}^{x} f(t)-f(x)D_n(t-x)\,dt\right| \\
&= \frac{1}{\pi}[f(x)-f(x-\delta)]\left|\int_{x-\delta}^{\xi} D_n(t-x)\,dt\right| < \varepsilon K_1 \\
|I_4| &= \frac{1}{\pi}\left|\int_{x+\delta}^{x+|pi} \frac{f(t)-f(x)}{2\sin\frac{1}{2}(t-x)}.\sin\left(n+\frac{1}{2}\right)(t-x)\,dt\right|
\end{aligned}$$

$\to 0$ uniformly with respect to x as $n \to \infty$. Hence the result.

Theorem 10 If $f(t) \in \mathcal{L}(-\pi\ \pi)$ and is periodic -2π and if

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin n)$$

and if $g(x) = \sum_0^x \left\{f(t) - \frac{1}{2}a_0\right\} dt$ then $g(x)$ is periodic 2π and $\sum_1^\infty \frac{b_n}{n} + \sum_1^\infty \frac{a_n \sin nx - b_n \cos nx}{n}$ converges uniformly to $g(x)$ in $[-\pi\ \pi]$.

Proof

$$
\begin{aligned}
g(x+2\pi) - g(x) &= \int_x^{x+2\pi} \{f(t) - \frac{1}{2}a_0\}\, dt \\
&= \int_x^{x+2\pi} f(t)\, dt - \pi a_0 \\
&= \int_{-\pi}^{\pi} f(t)\, dt - \pi a_0 = 0 \\
g(x) &\sim \frac{1}{2}A_0 + \sum_1^\infty A_n \cos nx + B_n \sin nx \\
n \geq 1, A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(t) \cos nt\, dt \\
&= \frac{1}{\pi}\left[g(t)\frac{\sin nt}{n}\right]_{-\pi}^{\pi} - \frac{1}{\pi n}\int_{-\pi}^{\pi} (f(t) - \frac{1}{2}a_0) \sin nt\, dt \\
&= -\frac{b_n}{n}
\end{aligned}
$$

Similarly $B_n = +\frac{a_n}{n}$

$$\int_0^x (f(t) - \frac{1}{2}a_0)\, dt = C + \sum_{n=1}^\infty \left(\frac{a_n}{n}\sin nx - \frac{b_n}{n}\cos nx\right)$$

putting $x = 0$ $\;C = \sum_{n=1}^\infty \frac{b_n}{n}$.

Theorem 11 If

(i) $f \in \mathcal{L}(-\pi\ \pi)$ and is periodic - 2π

(ii) $|f(t)| \leq M$ in $(x-\delta, x+\delta)$ $0 < \delta < \pi$

Then $|S_n(x)| \leq \frac{2}{\pi}M \log m + \frac{2M}{\pi} + O(1)$ as $m \to \infty$.

Proof

$$
\begin{aligned}
\frac{\pi}{2}S_n(x) &= \int_0^\pi \phi_x(t)\frac{\sin nt}{t}\, dt + O(1) \\
&= \int_0^\delta + \int_\delta^\pi + O(1) \\
&= \int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^\delta + \int_\delta^\pi + O(1) = I_1 + I_2 + I_3 + O(1)
\end{aligned}
$$

$$\left|\frac{\sin nt}{t}\right| \le \begin{cases} n & t \le \frac{1}{n} \\ \frac{1}{t} & t > \frac{1}{n} \end{cases}$$

$$\text{therefore } |I_1| \le \int_0^{\frac{1}{n}} |\phi_x(t)| \left|\frac{\sin nt}{t}\right| dt$$

$$\le M \int_0^{\frac{1}{n}} n\, dt = M$$

$$|I_2| \le \int_{\frac{1}{n}}^{\delta} |\phi_x(t)| \left|\frac{\sin nt}{t}\right| dt \le M \int_{\frac{1}{n}}^{\delta} \frac{1}{t}\, dt$$

$$= M(\log n + \log \delta) < M \log n$$

choosing $\delta < 1$ $I_3 \to 0$ as $n \to \infty$ by Riemann Lebesgue Theorem therefore

$$|S_n(x)| \le \frac{2}{\pi} M \log n + \frac{2M}{\pi} + O(1) \text{ as } n \to \infty.$$

Theorem 12 **(i)** $f(t) \in \mathcal{L}(-\pi\ \pi)$ and is periodic - 2π

(ii) $\phi_x(0+)$ exists

then $|S_n(x)| = O(\log n)$ as $n \to \infty$.

Proof

$$\frac{\pi}{2} S_n(x) = \int_0^{\pi} \phi_x(t) \frac{\sin nt}{t}\, dt + o(1)$$

If $\varepsilon > 0 \exists \eta$ with $o < \eta < 1 || \phi_x(t) - \phi_x(0+)| < \varepsilon$ in $(0, \eta)$.

$$\frac{\pi}{2}(S_n(x) - \phi_x((0+)) = \int_0^{\pi} [\phi_x(t) - \phi_x(0+)] \frac{\sin nt}{t} + o(1)$$

$$= \int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\eta} + \int_{\eta}^{\pi} + o(1)$$

$$= I_1 + I_2 + I_3 + o(1)$$

As in Theorem 11 $|i_1| \le \varepsilon$ $|I_2| \le \varepsilon(\log n + \log \eta) < \varepsilon \log n$ $|I_3| \to 0$ as $n \to \infty$ therefore $\frac{|S_n(X)|}{\log n} \to 0$ as $n \to \infty$.

Summability by Cesàro's 1st mean, summability (C,1) We have the result, due to Cauchy, that If $S_n \to S$ as $n \to \infty$ then $\frac{S_0 + S_1 + \ldots + S_n}{n+1} \to S$ as $n \to \infty$.

If $S_n = \sum_{r=1}^{n} a_r$ and if $\frac{S_0 + S_1 + \ldots + S_n}{n+1} = \sigma_n \to S$ as $n \to \infty$ we say $\sum_0^{\infty} a_r$ is summable $(C, 1)$ to S.

If $\sum_{r=0}^{\infty}$ converges it is summable $(C,1)$ to S, but the converse is not necessarily true.

Examples 1. $1-1+1-1+\ldots$ is summable $(C,1)$ to $\frac{1}{2}$ but is not convergent.

2.

$$\begin{aligned} S_n &= \frac{1}{2}+\cos\theta+\cos 2\theta+\ldots+\cos n\theta \;\; 0<\theta<2\pi \\ S_n &= \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta} \end{aligned}$$

$$\left|\frac{S_) + S_1+\ldots+S_n}{n+1}\right| = \left|\frac{1-\cos(n+1)\theta}{\left(2\sin\frac{1}{2}\theta\right)^2}\frac{1}{n+1}\right| \leq \frac{1}{n+1}\frac{2}{\left(2\sin\frac{1}{2}\theta\right)^2}$$

$\to 0$ as $n\to\infty$. Therefore $\frac{1}{2}+\sum_1^{\infty}\cos\nu\theta$ is summable $(C,1)$ to) for all $\theta|0<\theta<2\pi$.

But suppose $\sin\left(n+\frac{1}{2}\right)\theta \to S$ ads $n\to\infty$ therefore $\sin\left(n-\frac{1}{2}\right)\theta \to S$ as $n\to\infty$.

Therefore $\sin\left(n+\frac{1}{2}\right)\theta - \sin\left(n-\frac{1}{2}\right)\theta \to 0$ as $n\to\infty$

Therefore $2\cos n\theta\sin\frac{1}{2}\theta \to 0$ as $n\to\infty$. $\sin\frac{1}{2}\theta \neq 0$.

Therefore $\cos n\theta\to 0$ as $n\to\infty$ therefore $\cos 2n\theta \to 0$ as $n\to\infty$ but $\cos 2n\theta = 2\cos^2 n\theta - 1 \Rightarrow 0=-1$ which is a contradiction therefore $\frac{1}{2}\sum_{n=1}^{\infty}\cos\nu\theta$ does not converge.

Fejér's Integral

$$\begin{aligned} S_n(x) &= \frac{2}{\pi}\int_0^{\pi}\phi_x(t)D_n(t)\,dt \\ \text{therefore } \frac{S_0+\ldots S_n}{n+1} &= \sigma_n(x) \\ &= \frac{2}{\pi}\int_0^{\pi}\phi_x(t)\frac{D_0+\ldots+D_n}{n+1}\,dt \\ \sum_{\nu=0}^{n} D_\nu t &= \sum_{\nu=0}^{n}\frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} \\ &= \sum_{\nu=0}^{n}\frac{\cos\nu t-\cos(\nu+1)t}{\left(2\sin\frac{1}{2}\right)^2} \end{aligned}$$

$$\begin{aligned}
&= \frac{1-\cos(n+1)t}{\left(2\sin\frac{1}{2}\right)^2} \\
\text{Fejér's Kernel} &\equiv K_n(t) \\
&= \frac{D_0+\ldots+D_n}{n+1} \\
&= \frac{1}{n+1}\frac{1-\cos(n+1)t}{\left(2\sin\frac{1}{2}t\right)^2} \\
&= \frac{1}{2}\left(\frac{\sin\frac{n+1}{2}t}{\sin\frac{1}{2}t}\right)^2\frac{1}{n+1} \\
\sigma_n &= \frac{2}{\pi}\int_0^\pi \phi_x(t)K_n(t)\,dt.
\end{aligned}$$

Now if $f(t)=1$ for all t then $\phi_x(t)=1$ for all t, $S_n(t)=1$ for all t and $\sigma_n(t)=1$ for all t therefore

$$\sigma_n(x)-S=\frac{2}{\pi}\int_0^\pi(\phi_x(t)_S)K_n(t)\,dt$$

Theorem 13 Fejér's Theorem If $f\in\mathcal{L}(-\pi\ \pi)$ and is periodic 2π, and if $\phi_x(0+)$ exists then the Fourier series of $f(t)$ is summable $(C,1)$ for $t=x$ to $\phi_x(0+)$

Proof Let $\varepsilon>0$. Choose $\delta|0<\delta\le\pi$, and $|\phi_x(t)-\phi_x(0+)|<\frac{1}{2}\varepsilon$ in $0,\ \delta)$

$$\begin{aligned}
|I_1| &\le \frac{2}{\pi}\int_0^\delta|\phi_x(t)-\phi_x(0+)||K_n(t)|\,dt \\
&\le \frac{2}{\pi}\frac{\varepsilon}{2}\int_0^\delta K_n(t)\,dt \text{ as } K_n(t)\ge 0 \\
&\le \frac{2}{\pi}\frac{\varepsilon}{2}\int_0^\pi K_n(t)\,dt=\frac{\varepsilon}{2} \text{ for all } n\ge 0 \\
|I_2| &\le \frac{2}{\pi}\int_\delta^\pi|\phi_x(t)-\phi_x(0+)|\frac{1-\cos(n+1)t}{(n+1)4\sin^2\frac{1}{2}t} \\
&\le \frac{2}{\pi}\frac{1}{n+1}\int_\delta^\pi|\phi_x(t)-\phi_x(0+)|\frac{2}{4\sin^2\frac{1}{2}t}\,dt \\
&= \frac{C(\delta)}{n+1}<\frac{\varepsilon}{2}
\end{aligned}$$

if n is sufficiently large. Hence the result.

Corollary 1 If $f(x+)$ and $f(x-)$ exist, the Fourier series is summable $(C,1)$ to $\frac{f(x+)+f(x-)}{2}$.

Corollary 2 If $f(t)$ is continuous at x, the Fourier series is summable $(C,1)$ to $f(x)$.

Answer to Problem 1 The Fourier series converges to $\phi(x)$ therefore the Fourier series is summable $(C,1)$ to $\phi(x)$.

But since $f(t)$ continuous the Fourier series is summable $C,1)$ to $f(x)$ by Fejér's theorem therefore $f(x) = \phi(x)$.

Answer to Problem 2 The Fourier series of $f(x)$ is summable $(C,1)$ to $f(x)$.

The Fourier series of $\phi(x))$ is summable $(C,1)$ to $\phi(x)$.

If f and ϕ have the same Fourier series $f(x) = \phi(x)$.

Problem 4 If $f(t)$ is continuous and its Fourier series is convergent is its sum $f(t)$.

Theorem 14 If $f \in \mathcal{L}(-\pi\ \pi)$ and is periodic 2π then if the Fourier series converges at $t = x$, and $f(t)$ is continuous at $t = x$, its sum is $f(x)$.

Proof Since the Fourier series is convergent at $t = x$ it has sum S therefore the Fourier series is summable $(C,1)$ to S ate $t = x$ therefore $S = f(x)$.

Uniform Summability $\sum a_n(x)$ is summable $(C,1)$ uniformly in $[a\ b]$ to $\sigma(x)$ if

$$\sigma_n(x) = \frac{S_0(x) + \ldots + S_n(x)}{n+1} \to \sigma(x)$$

uniformly with respect to x in $[a\ b]$ as $n \to \infty$.

Theorem 15 Localisation Property If $f(4) \in \mathcal{L}(-\pi\ \pi)$ and is periodic 2π then

(i) the summability $(C,1)$ of the Fourier series at $t = x$ depends only on $f(t)$ in $x - \delta, x + \delta)$ $\delta > 0$.

(ii) the uniform summability $(C,1)$ of the Fourier series in $[a\ b]$ depends only on $f(t)$ in $(a - \delta, b + \delta\ \delta > 0$.

Proof (i)

$$\sigma_n(x) = \frac{2}{\pi}\int_0^{\pi} \phi_x(t)K_n(t)\,dt$$

For $0 < \delta < \pi$

$$\begin{aligned}
\left|\int_\delta^{\pi} \phi_x(t)K_n(t)\,dt\right| &= \frac{1}{n+1}\left|\int_\delta^{\pi} \phi_x(t)\frac{1-\cos(n+1)t}{\left(2\sin\frac{1}{2}t\right)^2}\,dt\right| \\
&\leq \frac{1}{n+1}\int_\delta^{\pi} |\phi_x(t)|\frac{2}{\left(2\sin\frac{1}{2}\right)^2}\,dt \\
&= \frac{C(\delta)}{n+1} \to 0 \text{ as } n\to\infty
\end{aligned}$$

(ii)

$$\begin{aligned}
\sigma_n(x) &= \frac{2}{\pi}\int_0^{\pi} \phi_x(t)K_n(t)\,dt \\
&= \frac{1}{\pi}\int_0^{\pi} f(x+t)+f(x-t)K_n(t)\,dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)K_n(t)\,dt \\
&= \frac{1}{\pi}\int_{x-\pi}^{x+\pi} f(t)K_n(t-x)\,dt \\
&= \frac{1}{\pi}\int_{x-\pi}^{x-\delta} + \frac{1}{\pi}\int_{x-\delta}^{x+\delta} + \frac{1}{\pi}\int_{x+\delta}^{x+\delta} \\
&= I_1 + I_2 + I_3 \\
|I_3| &= \left|\frac{1}{\pi}\int_{x+\delta}^{x+\pi} f(t)K_)n(t-x)\,dt\right| \\
&\leq \frac{1}{n+1}\frac{1}{\pi}\int_{x+\delta}^{x+\pi} |f(t)|\frac{2}{\left[2\sin\frac{1}{2}(t-x)\right]^2}\,dt \\
&\leq \frac{2}{n+1}\frac{1}{\pi}\frac{1}{\left(2\sin\frac{1}{2}\delta\right)^2}\int_{x+\delta}^{x+\pi} |f(t)|\,dt \\
&\leq \frac{2}{n+1}\frac{1}{\pi}\frac{1}{\left(2\sin\frac{1}{2}\delta\right)^2}\int_{-\pi}^{\pi} |f(t)|\,dt = \frac{C(\delta)}{n+1} \to 0
\end{aligned}$$

uniformly in any interval and so in the whole range. Similarly $I_1 \to 0$ uniformly in the whole range.

Theorem 16 (Modified form of $K_{n-1}(t)$) The necessary and sufficient condition for the Fourier series of $f(t)$ to be summed $(C,1)$ to S is

$$\frac{1}{n}\int_0^\pi (\phi_x(t)-S)\left(\frac{\sin\frac{1}{2}nt}{t}\right)^2 dt \to 0 \text{ as } n\to\infty$$

Proof

$$\begin{aligned}
K_{n-1}(t) &= \frac{1}{n}\frac{1-\cos nt}{\left(2\sin\frac{1}{2}t\right)^2} \\
&= \frac{2}{n}\frac{\sin^2\frac{1}{2}n(t)}{\left(2\sin\frac{1}{2}t\right)^2} \\
I &= \frac{1}{n}\left|\int_0^\pi [\phi_x(t)-S]\left[\left(\frac{\sin\frac{1}{2}nt}{2\sin\frac{1}{2}t}\right)^2 - \left(\frac{\sin\frac{1}{2}nt}{t}\right)^2\right] dt\right| \\
&\le \int_0^\pi |\phi_x(t)-S|\left[\frac{1}{\left(2\sin\frac{1}{2}t\right)^2} - \frac{1}{t^2}\right] dt
\end{aligned}$$

write $g(t) = \frac{1}{\left(2\sin\frac{1}{2}t\right)^2} - \frac{1}{t^2}$.

$g(t)$ continuous $0 < t \le \pi$ and tends to a limit as $t \to 0$. Define $g(0)$ by continuity then $g(t)$ bounded in $[0\ \pi]$

$$I \le \frac{1}{n}A\int_0^\pi |\phi_x(t)-S|\,dt = O\left(\frac{1}{n}\right)$$

Hence the result.

The Lebesgue Set Lebesgue showed that if $g(t) \in \mathcal{L}(a\ b)$ and $\phi(x) = \int_a^x g(t)\,dt$ $x \in [a\ b]$ then $\exists \phi'(x) = g(x)$ p.p in $(a\ b)$. He then generalised it to the following result.

Theorem 17 If $f \in \mathcal{L}(a\ b)$ $\lim_{h\to 0}\frac{1}{h}\int_x^{x+h}|f(t)-\alpha|\,dt = |f(x)-\alpha|$ for all real α, except when x belongs to a set of measure 0 (independent of α).

Proof For a fixed α $\lim_{h\to 0}\frac{1}{h}\int_x^{x+h}|f(t)-\alpha|\,dt = |f(x)-\alpha|$ for all $x \in [a\ b)$ outside a set ξ_α of measure zero.

Let $\{\alpha_\nu\}$ be an enumeration of the rational numbers.

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h}|f(t)-\alpha_\nu|\,dt = |f(t)-\alpha_\nu|$$

for all α_ν, and all x outside $\xi = \cup_{\nu=1}^{\infty}\xi_{\alpha_n u}$ which is null.

Let β be a real number.

$$||f(x)-\beta|-|f(x)-\alpha_\nu|| \le |(f(x)-\beta)-(f(x)-\alpha_\nu)| = |\beta-\alpha_\nu|$$

Therefore

$$\left|\frac{1}{h}\int_x^{x+h}|f(t)-\beta|\,dt - \frac{1}{h}\int_x^{x+h}|f(t)-\alpha_\nu|\,dt\right| \le |\beta-\alpha_\nu|$$

for all x outside ξ we have

$$\begin{aligned}
&\left|\frac{1}{h}\int_x^{x+h}|f(t)-\beta|\,dt - |f(t)-\beta|\right| \\
\le\ &\left|\frac{1}{h}\int_x^{x+h}\int_x^{x+h}|f(t)-\beta|\,dt - \frac{1}{h}\int_x^{x+h}|f(t)-\alpha_\nu|\,dt\right| \\
&+\left|\frac{1}{h}\int_x^{x+h}|f(t)-\alpha_\nu|\,dt - |f(t)-\alpha_\nu|\,dt\right| \\
&+||f(t)-\alpha_\nu|-|f(t)-\beta|| \\
\le\ &|\beta-\alpha_\nu| + \left|\frac{1}{h}\int_x^{x+h}|f(t)-\alpha_\nu|\,dt - |f(t)-\alpha_n u|\right| + |\beta-\alpha_\nu|
\end{aligned}$$

which may be made as small as we please, by choice first of α_ν and then of h. Hence the result.

Corollary

$$\int_0^h |f(x+t)-f(x)|\,dt = o(h) \text{ as } h\to 0$$

for almost all x. The set where this holds is called the Lebesgue set.

Theorem 18 (Fejér Lebesgue) If $f(t)\in\mathcal{L}(-\pi\ \pi)$ and is periodic 2π and if $\int_0^h|\phi_x(t)-S|\,dt = o(h)$ as $h\to 0+$ then the Fourier series of $f(t)$ is summable $(C,1)$ to S at $t=x$.

Proof

$$\frac{1}{n}\int_0^\pi [\phi_x(t)-S]\left(\frac{\sin\frac{1}{2}nt}{t}\right)^2 dt = \sigma_n(x)+o(1) \qquad (3)$$

Let $\varepsilon > 0$. $\exists\delta|0<\delta<\pi$, and

$$\begin{aligned}
0 \le \Phi(t) &= \int_0^t |\phi_x(u)-S|\,du < \varepsilon t \text{ in } 90\ \delta] \\
\frac{1}{n}\left\{\frac{\sin\frac{1}{2}nt}{t}\right\}^2 &\le \begin{cases} n & 0<t\le\frac{1}{n} \\ \frac{1}{nt^2} & \frac{1}{n}\le t\le\pi \end{cases} \\
(3) &= \frac{1}{n}\left\{\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \in_\delta^\pi\right\} \\
&= I_1+I_2+I_3 \text{ taking } n > \frac{1}{\delta} \\
|I_3| &\le \frac{1}{n}\int_\delta^\pi |\phi_x(t)-S|\frac{1}{t^2}\,dt = \frac{C(\delta)}{n} \\
|I_1| &\le n\int_0^{\frac{1}{n}} |\phi_x(t)-S|\,dt \to 0 \text{ as } n\to\infty \text{ by hypothesis} \\
|I_2| &\le \frac{1}{n}\int_{\frac{1}{n}}^{\delta} |\phi_x(t)-S|\,\frac{dt}{t^2} \\
&= \frac{1}{n}\left[\frac{\Phi(t)}{t^2}\right]_{\frac{1}{n}}^{\delta} + \frac{2}{n}\int_{\frac{1}{n}}^{\delta}\frac{\Phi(t)}{t^3}\,dt \\
&\le \frac{1}{n}C(\delta) - n\Phi\left(\frac{1}{n}\right) + \frac{2\varepsilon}{n}\int_{\frac{1}{n}}^{\delta}\frac{dt}{t^2} \\
&\le \frac{1}{n}C(\delta) + \frac{2\varepsilon}{n}\int_{\frac{1}{n}}^{\infty}\frac{dt}{t^2} \\
&= \frac{C(\delta)}{n} + 2\varepsilon
\end{aligned}$$

Hence the result.

Corollary

$$\frac{1}{h}\int_0^h |\phi_x(t)-f(x)|\,dt \le \frac{1}{h}\int_0^h \frac{|f(x+t)-f(x)|}{2}\,dt + \frac{1}{h}\int_0^h \frac{|f(x-t)-f(x)|}{2}\,dt$$

$\to 0$ in the Lebesgue set.

Hence the Fourier series of $f(t)$ is summable $(C,1)$ to $f(x)$ in the Lebesgue set.

Theorem 19 The necessary and sufficient conditions for $m \leq \sigma_n(x) \leq M$ for all n and all x is $m \leq f(t) \leq M$ p.p. in $-\pi\ \pi]$

Proof (i) Necessity: Since $f(t) \in \mathcal{L}(-\pi\ \pi)$, for all x in the Lebesgue set $\sigma_n(x) \to f(x)$ as $n \to \infty$.

By hypothesis $m \leq \sigma_n(x) \leq M$ therefore $m \leq \underline{\lim}\sigma_n(x) \leq \overline{\lim}\sigma_n(x) \leq M$ therefore for almost all x, $\overline{\lim} = \underline{\lim} = f(x)$ therefore $m \leq f(x) \leq M$ p.p in $[-\pi\ \pi]$.

(ii) Sufficiency:

$$\sigma_n(x) - M = \frac{1}{2\pi}\int_0^\pi \frac{\phi_x(t) - M}{n+1}\left\{\frac{\sin\frac{n+1}{2}t}{\sin\frac{1}{2}t}\right\}^2 dt$$

Since $f(t) \leq M$ p.p. in $[-\pi\ \pi]\phi_x(t) \leq M$ p.p in $[0\pi]$ therefore the integrand is ≤ 0p.p therefore the integral is ≤ 0 therefore $\sigma_n(x) \leq M$. Similarly $\sigma_n(x) \geq m$.

Theorem 20 Uniqueness theorem for Fourier Series If $f \in \mathcal{L}(-\pi\ \pi)$ $g \in \mathcal{L}(-\pi\ \pi) f(t) \sim \frac{1}{2}a_0 + \sum(\ )$ $g(t) \sim \frac{1}{2}a_0 + \sum(\ )$ then $f(t) = g(t)$ p.p. in $(-\pi\ \pi)$.

Proof The Fourier series of $f(t)$ is summable $(C, 1)$ p.p. to $f(t)$. The Fourier series of $g(t)$ is summable $(C, 1)$ to $g(t)$.

Theorem 21 If $f(t) \in \mathcal{L}(-\pi\ \pi)$ and it's Fourier coefficients are all zero the $f(t) = 0$ p.p in $(-\pi\ \pi)$.

Proof From Theorem 20 with $g(t) = 0$. Suppose $f(t)$ is even and $|f(t)||leq1$.

$$\begin{aligned} S_n &= S_n(0) = \frac{2}{\pi}\int_0^\pi f(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}\,dt \\ |S_n| &\leq \frac{2}{\pi}\int_0^\pi \frac{|\sin\left(n+\frac{1}{2}\right)t|}{2\sin\frac{1}{2}t}\,dt \end{aligned}$$

$$\text{If } f(t) = \chi_n(t) = \begin{cases} 1 & \text{if } \sin\left(n+\frac{1}{2}\right)t > o \\ -1 & \text{if } \sin\left(n+\frac{1}{2}\right)t \leq 0 \end{cases} \text{ in } [-\ \pi]$$

$$S_n(0) = \frac{2}{\pi}\int_0^\pi \frac{|\sin\left(n+\frac{1}{2}\right)|}{2\sin\frac{1}{2}t}\,dt.$$

Let $L_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi \frac{\left|\sin\left(n+\frac{1}{2}\right)t\right|}{2\sin\frac{1}{2}t}\,dt$

We can show that $|L_n - \frac{4}{\pi^2}\log n| < K$ for all n, and some K, and $L_n \sim \frac{4}{\pi^2}\log n$ as $n \to \infty$.

$$\begin{aligned} & \left|\frac{2}{\pi}\int_0^\pi \frac{|\sin\left(n+\frac{1}{2}\right)t|}{2\sin\frac{1}{2}t}\,dt - \frac{2}{\pi}\int_0^\pi \frac{|\sin nt|}{2\tan\frac{1}{2}t}\,dt\right| \\ \leq\ & \frac{2}{\pi}\int_0^\pi \frac{|\sin\left(n+\frac{1}{2}\right)t - \sin nt\cos\frac{1}{2}t|}{2\sin\frac{1}{2}t}\,dt \\ =\ & \frac{2}{\pi}\int_0^\pi \frac{|\cos nt|}{2} \leq \frac{2}{\pi}\int_0^\pi \frac{1}{2}\,dt = 1 \end{aligned}$$

Now

$$\begin{aligned} & \left|\frac{2}{\pi}\int_0^\pi \left(\frac{|\sin nt|}{2\tan\frac{1}{2}t} - \frac{|\sin nt|}{t}\right)dt\right| \\ \leq\ & \frac{2}{\pi}\int_0^\pi \left|\frac{1}{t} - \frac{1}{2\tan\frac{1}{2}t}\right| dt = G \end{aligned}$$

therefore $$\left|L_n - \frac{2}{\pi}\int_0^\pi \frac{|\sin nt|}{t}\,dt\right| \leq G+1$$

$$\begin{aligned} \int_0^\pi \frac{|\sin nt|}{t}\,dt &= \int_0^n \pi\frac{|\sin u|}{u}\,du \\ &= \sum_{\nu=0}^{n-1}\int_{\nu\pi}^{(\nu+1)\pi} \frac{|\sin u|}{u}\,du \\ &= \sum_{\nu=0}^{n-1}\int_0^\pi \frac{|\sin u|}{u+\nu\pi}\,du \\ &\geq \sum_{\nu=0}^{n-1}\int_0^\pi \sin u\,du\frac{1}{(\nu+1)\pi} \text{ as } \frac{1}{\nu\pi} \geq \frac{1}{u+\nu\pi} \geq \frac{1}{(\nu+1)\pi} \\ &= \sum_{\nu=0}^{n-1}\frac{2}{(\nu+1)\pi} \\ &\geq \frac{2}{\pi}\sum_{\nu=0}^{n-1}\sum_{\nu+1}^{\nu+2}\frac{dt}{t} \\ &\geq \frac{2}{\pi}\log n \end{aligned}$$

Similarly

$$\sum_{\nu=0}^{n-1}\int_0^\pi \frac{\sin u}{u+\nu n} \le \int_0^\pi \frac{\sin u}{u}\,du + \int_0^\pi \frac{\sin u}{u+\pi}\,du + \frac{2}{\pi}\log n$$

Therefore

$$\left|L_n - \frac{4}{\pi^2}\log n\right| < K$$

Hence $\overline{\lim}\frac{S_n(x)}{\log n} \le \frac{4}{\pi^2}$.

Theorem 22 There is an even continuous periodic function of period 2π whose Fourier series diverges for $t = 0$.

Proof

$$S_n(0) \ge \frac{2}{\pi}\int_0^\pi f(t)\frac{\sin nt}{t}\,dt - C$$

Write $S_n = \frac{2}{\pi}\int_0^\pi f(t)\frac{\sin nt}{t}\,dt$ then $S_n(0) \ge \frac{2}{\pi}S_n - C$.

We shall construct $F(t)$ so that $\overline{\lim}S_n = +\infty$. Consider $a_\nu > 0 | \sum_1^\infty = 1$.

Consider integers $1 < n_1 < n_2 < n_3 < \dots$

$f(t) = \sum_{\nu=1}^\infty a_\nu \sin n_\nu t$ in $[0\ \pi]$, even and periodic 2π.

$f(t)$ is continuous as the series converges uniformly with respect to t.

$$\begin{aligned} S_n &= \int_0^\pi f(t)\frac{\sin nt}{t}\,dt \\ &= \int_0^\pi \sum_{\nu=1}^\infty a_\nu \sin n_\nu t.\frac{\sin nt}{t}\,dt \\ &= \sum_{\nu=1}^\infty a_\nu \int_0^\pi \frac{\sin n_\nu t \sin nt}{t}\,dt \end{aligned}$$

Choose $n_1 | \alpha_2 1 \int_0^\pi \frac{\sin^2 n_1 t}{t} > 2$

$$S_{n_1} \ge \alpha_1 \int_0^\pi \frac{\sin^2 n_1 t}{t}\,dt - \sum_{\nu=2}^\infty \alpha_\nu \left|\sum_0^\pi \frac{\sin n_1 t}{t}\sin n_\nu t\,dt\right|$$

Choose $M_1 | \left|\int_0^\pi \frac{\sin n_1 t}{t}\sin mt\,dt\right| < 1$ for $m \ge M_1$ (possible be Riemann-Lebesgue Theorem) therefore provided $n_2 \ge M_1$

$$\left|\int_0^\pi \frac{\sin n_\nu t \sin n_1 t}{t}\,dt\right| < 1 \ \nu = 2, 3, \ldots$$

therefore $S_{n_1} \geq 2 - \sum_{\nu=2}^{\infty} \alpha_\nu \geq 1$.

Choose $n_2 \geq M_1 | \alpha_2 \int_0^\pi \frac{\sin^2 n_2 t}{f}\,dt > 3$.

$$S_{n_2} \geq \alpha_2 \int_0^\pi \frac{\sin^2 n_2 t}{t}\,dt - \left(\sum_{\nu<2} + \sum_{\nu>2}\right)\left|\alpha_2 \int_0^\pi \frac{\sin n_\nu t \sin n_2 t}{t}\,dt\right|$$

$$\left|\int_0^\pi \frac{\sin n_1 t \sin n_2 t}{t}\,dt\right| < 1 \text{ as } n_2 \geq M_1.$$

Choose $M_2 | \left|\int_0^\pi \frac{\sin n_2 t}{t} \sin mt\,dt\right| < 1$ for $m \geq M_2$ therefore provided $n_3 \geq M_2$

$$\left|\int_0^\pi \frac{\sin n_\nu t \sin n_2 t}{t}\,dt\right| < 1 \ \nu = 3, 4, \ldots$$

therefore $S_{n_2} \geq 3 - \sum_{\nu \neq 2} \alpha_\nu .1 \geq 2$.

Suppose $1 \leq n_1 < M_1 \leq n_2 < M_2 \leq \ldots \leq n_{\mu-1} < M_{\mu-1}$ have all been chosen such that $S_{n_{\mu-1}} > \mu - 1$ provided $n_\mu \geq M_{\mu-1}$,

Choose $n_\nu | \alpha_\mu \int_0^\pi \frac{\sin^2 n_\mu t}{t}\,dt > \mu + 1$

$$\left|\int_0^\pi \frac{\sin n_\nu t \sin n_\mu t}{t}\right| < 1 \text{ for } \nu = 1, 2, \ldots, \mu - 1 \text{ as} n_\mu \geq M_{\mu-1}$$

Choose $M_\mu | \left|\int_0^\pi \frac{\sin n_\mu t \sin mt}{t}\,dt\right| < 1$ provided $m \geq M_\mu$.

Therefore provided $n_{\mu+1} \geq M_\mu$

$$\begin{aligned} S_{n_\mu} &\geq \alpha_\nu \int_0^\pi \frac{\sin^2 n_\mu t}{t}\,dt - \sum_{\nu \neq \mu} \alpha_\nu \left|\int_0^\pi \frac{\sin n_\nu t \sin n_\mu t}{t}\,dt\right| \\ &\geq \mu + 1 - 1 = \mu \end{aligned}$$

Lemma 1 If $f_N(t) = \sin\left(N + \frac{1}{2}\right)t$ in $[0\ \pi]$ even and periodic, N a positive integer. Then

(i) $S_n(0) > 0$

(ii) $S_N(0) > \frac{1}{\pi} \log N - C$

Proof (i)

$$a_n = \frac{1}{\pi}\left(\frac{1}{N+n+\frac{1}{2}} + \frac{1}{N-n+\frac{1}{2}}\right) = \frac{1}{\pi}\frac{2N+\frac{1}{2}}{\left(N+\frac{1}{2}\right)^2 - n^2}$$

$S_n(0)$ increasing and > 0 in $[0\ N]$ and decreasing in $[N+1\ \infty)$, also $S_\infty(0) = 0$ and the function is continuous and B.V therefore $S_n(0) > 0$ for all n.

(ii) See problems.

Theorem 23 If $\psi(n)$ is any decreasing sequence tending to zero as $n \to \infty$, $\exists$ a continuous even periodic function of period 2π whose Fourier series

(i) diverges at 0

(ii) $s_{n_\nu}(0) > \log n_\nu \psi(n_\nu)$ for a sequence $n_1 < n_2 < \ldots$.

Proof Let $\alpha_j = \frac{1}{j^2},\ f_n(t) = \sin\left(n+\frac{1}{2}\right)t\ [0\ \pi]$ even and periodic.

$$\begin{aligned}
f(t) &= \sum_{j=1}^{\infty} \alpha_j f_{n_j}(t) \\
&= \sum_{j=1}^{\infty} \frac{\sin\left(n_j+\frac{1}{2}\right)t}{j^2} \\
S_{n_\nu}(0) &= \sum_{j=1}^{\infty} \frac{1}{j^2}\frac{2}{\pi}\int_0^\pi \frac{\sin\left(n_j+\frac{1}{2}\right)t \sin\left(n_\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}\,dt \\
&\geq \frac{1}{\nu^2}\frac{2}{\pi}\int_0^\pi \frac{\sin^2\left(n_\nu+\frac{1}{2}\right)}{2\sin\frac{1}{2}t}\,dt \\
&> \frac{1}{\pi}\frac{1}{\nu^2}\log n_\nu - C \\
&> \frac{1}{2\pi}\frac{1}{\nu^2}\log n_\nu
\end{aligned}$$

is ν sufficiently large.

Choose $n_\nu | \psi(n_\nu) < K\frac{1}{2\pi}.\frac{1}{\nu^2}$ (possible as $\psi \to 0$).

General Trigonometrical Series

Theorem 24 Cantor's Lemma If $a_n \cos nx + b_n \sin nx \to 0$ for all x in a set E of positive measure, then $a_n \to 0, b_n \to 0$ as $n \to \infty$.

Proof Suppose without loss of generality $\exists \subset [-\pi\ \pi]$

Assume $a_n^2 + b_n^2 \not\to 0$ $\exists \delta > 0$ and $\{n_\nu\} | a_{n_\nu}^2 + b_{n_\nu}^2 \geq \delta$.

Let $g_\nu(x) = \dfrac{(a_{n_\nu} \cos n_\nu x + b_{n_\nu} \sin n_\nu x)^2}{a_{n_\nu}^2 + b_{n_\nu}^2}$

$g_\nu(x) \to 0$ boundedly in E for

$$\begin{aligned} |g_n u(x)| &\leq \frac{a_{n_\nu} \cos n_\nu x + b_{n_\nu} \sin n_\nu x)^2}{\delta} \to 0,\ x \in E \\ |g_\nu(x)| &= \left(\frac{a_{n_\nu}}{\sqrt{a_{n_\nu}^2 + b_{n_\nu}^2}} \cos n_\nu x + \frac{b_{n_\nu}}{\sqrt{a_{n_\nu}^2 + b_{n_\nu}^2}} \sin n_n u x \right)^2 \\ &= \sin^2(n_\nu x + \alpha_\nu \leq 1 \end{aligned}$$

Therefore $\int_e g_\nu(x)\, dx \to 0$ as $\nu \to \infty$ by the Dom. Cgce. Theorem.

$$\begin{aligned} \int_E g_\nu(x)\, dx &= \int_E \frac{(a_{n_\nu} \cos n_\nu x + b_{n_\nu} \sin n_\nu x)^2}{a_{n_\nu}^2 + b_{n_\nu}^2}\, dx \\ &= \int_E \frac{a_{n_\nu}^2 \cos^2 n_\nu x + 2a_\nu b_\nu \sin n_\nu x \cos n_\nu x + b_{n_\nu}^2 \sin^2 n_\nu x}{a_{n_\nu}^2 + b_{n_\nu}^2}\, dx \\ &= \frac{1}{2} \int_E \frac{a_{n_\nu}^2 + b_{n_\nu}^2}{a_{n_\nu}^2 + b_{n_\nu}^2} + C \int_E \cos 2n_\nu x\, dx + C' \int_E \sin 2n_\nu x\, dx \\ &= I_1 + I_2 + I_3 \end{aligned}$$

$I_2, I_3 \to 0$ as $n \to \infty$ by Riemann-Lebesgue theorem, therefore $\int_E g_\nu(x)\, dx \to \frac{1}{2} mE$ as $n \to \infty$ which is a contradiction.

Theorem 25 If $\dfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + bb_n \sin nx$ converges to S and if $F(t) = \dfrac{1}{4}a_0 t^2 - \sum_1^\infty \dfrac{a_n}{n^2} \cos nt + \dfrac{b_n}{n^2} \sin nt$ converges in $|t - x| < \delta$.

$$\lim_{h \to 0} \frac{F(x+h) - 2F(x) + F(x-h)}{h^2} = S$$

Proof If $0 < h < \delta$ then we have

$$\begin{aligned}
& \frac{F(x+h) - 2F(x) + F(x-h)}{h^2} \\
= \; & \frac{1}{4}a_0^2 \frac{(x+h)^2 - 2x^2 + (x-h)^2}{h^2} \\
& -\frac{1}{h^2}\sum_{n=1}^{\infty}\left[\frac{a_n}{n^2}\{\cos n(x+h) - 2\cos nx + \cos n(x-h)\}\right. \\
& \left. + \frac{b_n}{n^2}\{\sin n(x+h) - 2\sin nx + \sin n(x-h)\}\right] \\
= \; & \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\left(\frac{\sin\frac{1}{2}nh}{\frac{1}{2}nh}\right)^2
\end{aligned}$$

$\to S$ as $h \to 0$.

Corollary Riemann's 1st Lemma If $a_n \to 0$ and $b_n \to 0$, and if $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ converges to S, $F(t) = \frac{1}{4}a_0 t^2 - \sum_{n=1}^{\infty} \frac{a_n}{n^2}\cos nt + \frac{b_n}{n^2}\sin nt$ exists for all t and $\exists \lim_{h\to 0} \frac{F(x+h) - 2F(x) + f(x-h)}{h^2} = S$.

Generalised Derivatives I First derivative $f'(x) = \lim_{h\to 0} fracf(x+h) - f(x)h$

Symmetric 1st derivative $D'f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h}$

[If $\exists f'(x) \Rightarrow D'f(x) = f'(x)$]

II Generalised 2nd derivative.

$$f_{(2)}(x) = \lim_{h\to 0} \frac{f(x+h) - f(x) - f'(x).h}{\frac{1}{2}h^2}$$

[If $\exists f''(x) \Rightarrow f_{(2)}(x) = f''(x)$]

III Symmetric generalised 2nd derivative.

$$D^2 f(x) = \lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

[If $\exists f''(x) \Rightarrow D^2 f(x) = f''(x)$]

Schwarz's Lemma If $D^2F(x) = 0$ in $(a\ b)$ and F is continuous in $[a\ b] \Rightarrow f(x) = Ax + B$ in $[a\ b]$.

Proof We prove: If $D^2F(x) \geq 0$ in $(a\ b)$, then

$$F(x) \leq \frac{F(a)(b-x)+F(b)(x-a)}{(b-a)}$$

in $[a\ b]$.

Let $L(x) = \dfrac{F(a)(b-x)+F(b)(x-a)}{b-a}$

$L''(x) = 0$ therefore $D^2L(x) = 0$.

Let $\phi(x) = F(x) - L(x)$.

$\phi(x)$ is continuous in $[a\ b]$ and $D^2\phi(x) = D^2F(x) \geq 0$.

$\phi(a) = \phi(b) = 0$ R.T.P. $\phi(c) \leq 0$ $a < c < b$.

Suppose $\phi(c) > 0$ $a < c < b$ therefore the upper bound of ϕ in $(a\ b)$ is > 0 and attained at ξ say.

Let $g(x) = -\frac{1}{2}\varepsilon(x-a)(b-x)$ $\ g''(x) = \varepsilon$.

Let $\psi(x) = \phi(x) + g(x)$.

$\psi(x)$ is continuous in $[a\ b]$ and

$$D^2\psi(x) = D^2\phi(x) + D^2g(x) \geq \varepsilon.$$

Choose $\varepsilon | \psi(c) > 0$ therefore the upper bound of ψ in $(a\ b)$ is > 0 and attained at ζ say.

$a < \zeta < b$ and $\psi(\zeta) \geq \psi(x)$ $x \in [a\ b]$ therefore

$$\frac{\psi(\zeta+h) - 2\psi(\zeta) + \psi(\zeta-h)}{h^2} \leq 0$$

therefore $D^2\psi(\zeta) \leq 0$ which is a contradiction.

Therefore $D^2F(x) \geq 0$ in $(a\ b)$

$$\Rightarrow F(x) \leq \frac{F(a)(b-x)+F(b)(x-a)}{(b-a)}$$

Therefore $D^2F(x) = 0$ in $(a\ b)$

$$\Rightarrow F(x) = \frac{F(a)(b-x)+F(b)(x-a)}{(b-a)}$$

Theorem 26 1st uniqueness theorem for Trig. Series If a Trigonometric series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ converges to zero for all x then $a_0 = a_n = b_n = 0$.

Proof

1. Since the trigonometric series converges for all x $(a_n \cos nx + b_n \sin nx) \to 0$ as $n \to \infty$ for all x in a set of positive measure.
2. By Cantor's Lemma $a_n \to 0$ $b_n \to 0$ as $n \to \infty$
3. Therefore $F(t) = \frac{1}{4}a_0 t^2 - \sim_{n=1}^{\infty} \frac{a_n}{n^2} \cos nt + \frac{b_n}{n^2} \sin nt$ converges uniformly with respect to t in any finite interval therefore $F(t)$ is continuous everywhere.
4. For each x $\frac{1}{2}a_0 + \sum_1^{\infty} a_n \cos nx + b_n \sin nx$ converges to zero therefore by Riemann's 1st Lemma $D^2F(x)$ exists and $=0$ for every x.
5. By Schwarz's Lemma, in an arbitrary finite interval $[a\ b]$, since F is continuous, it follows that $F(x)$ is linear in $(a\ b)$.
6. Since $F(x)$ is linear in every interval $[a\ b)$, $F(x)$ in linear in $-\infty,\ \infty)$
7. $$\frac{1}{4}a_0 x^2 - \sum_{n=1}^{\infty} \frac{a_n}{n^2} \cos nx + \frac{b_n}{n^2} \sin nx = Ax + B$$
   therefore
   $$\sum_{bn=1}^{\infty} \frac{a_n}{n^2} \cos nx + \frac{b_n}{n^2} \sin nx = \frac{1}{4}a_0 x^2 - Ax - B \quad -\infty < x < \infty$$
   But LHS is periodic - 2π and continuous in $[-\pi\ \pi]$ and therefore is bounded in $(-\infty\ \infty)$. Therefore $a_0 =)$ $A = 0$.
8. Therefore
   $$B = \sum_1^{\infty} \frac{a_n}{n^2} \cos nx + \frac{b_n}{n^2} \sin nx$$
   $$\frac{a_n}{n^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} -B \cos nx \, dx$$
   $$\frac{b_n}{n^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} -B \sin nx \, dx$$
   Therefore $a_n = b_n = 0$.

Theorem 27 2nd Uniqueness theorem for trig series If for all x with a finite number of exceptions $\frac{1}{2}a_0 + \sum_{n=1}^{i} nfty a_n \cos nx + b_n \sin nx$ converges to zero then $a_0 = 0$ $a_n = b_n = 0$ $n \geq 1$

Lemma Riemann's 2nd Lemma If a_n and $b_n \to 0$ as $n \to \infty$ and $F(t) = \frac{1}{4}a_0t^2 - \sum_{n=1}^{\infty} \frac{a_n}{n^2} \cos nt + \frac{b_n}{n^2} \sin nt$ then $\lim_{h \to 0} \frac{F(x+h) - 2F(x) + F(x-h)}{h} = 0$

Proof

$$\begin{aligned} & \frac{F(x+h) - 2F(x) + F(x-h)}{h} \\ = & \frac{1}{2}a_0 h + h \sum_{n=1}^{i} nfty(a_n \cos nx + b_n \sin nx) \left(\frac{\sin \frac{1}{2}nh}{\frac{1}{2}nh} \right)^2 \end{aligned}$$

If $\varepsilon > 0 \exists N || a_n \cos nx + b_n \sin nx| < \varepsilon n > N$.

Also $\sin^2 \frac{1}{2}nh \leq \left(\frac{1}{2}nh\right)^2$ for $n \leq \frac{1}{h}$.

Let A be the upper bound of $\left\{ \frac{1}{2}|a_0| |a_n \cos nx + b_n \sin nx| \right\}$.

Then

$$\begin{aligned} & \left| \frac{F(x+h) - 2F(x) + F(x-h)}{h} \right| \\ \leq & Ah + \sum_{n=1}^{N} Ah + h \sum_{N < n \leq \frac{1}{h}} \varepsilon + 4 \sum \frac{\varepsilon}{n^2 h} \\ \leq & A(N+1)h + \varepsilon + \frac{4\varepsilon}{h} \int_{\frac{1}{h}}^{\infty} \frac{du}{(u-1)^2} \\ = & A(N+1)h + \varepsilon + \frac{4\varepsilon}{1-h} \end{aligned}$$

this may be made as small as we please first by choice of ε, then by choice of h.

Proof of Theorem 27 Theorem 26 tells us that $F(t)$ is linear in the interval between any two of the exceptional points, and since F is continuous the straight lines in adjacent intervals join. Applying Riemann's 2nd Lemma at the exceptional point tells us that the slopes on both sides of this point are the same. Hence $F(t)$ is linear throughout $(-\infty\ \infty)$ and the result follows as in theorem 26.

Theorem 28 If a trigonometric series $\frac{1}{2}a_0 + \sum(a_n \cos nx + b_n \sin nx)$ converges for all x to $f(x)$, and f is bounded then the trigonometric series is the Fourier series of $f(x)$.

Lemma 1 $F(x)$ continuous in $[a\ b]$ and $D^2F(x) \geq 0$ in $[a\ b] \Rightarrow F(x)$ convex in $[a\ b]$ i.e. if $a \leq \alpha < \gamma < \beta \leq b$

$$F(\gamma) \leq \frac{F(\alpha)(\beta-\gamma) + F(\beta)(\gamma-\alpha)}{\beta-\alpha}$$

In particular $\beta = x+h$ $\alpha = x-h$ $\gamma = x$ gives

$$F(x) \leq \frac{F(x+h) + F(x-h)}{2}$$

Therefore

$$\Delta_h^2 = F(x+h) + F(x-h) - 2F(x) \geq 0$$

i.e. as $D^2F(x) = \lim_{h\to 0} \dfrac{\Delta_h^2 F(x)}{h^2} \geq 0$ in $(a\ b) \Rightarrow \frac{1}{h^2}\Delta_h^2 F(x) \geq 0$ $a < x < b$ if h is sufficiently small.

Lemma 2 If $F(x)$ is continuous in $[a\ b]$ and if $m \leq D^2F(x) \leq M$ in $(a\ b)$ and if $m \leq D^2F(x) \leq M$ in $(a\ b)$ then $m \leq \dfrac{\Delta_h^2 F(x)}{h^2} \leq M$ $D^2F(x) - m \geq 0$

Therefore

$$D^2\left(F(x) - \frac{1}{2}mx^2\right) \geq 0 \Rightarrow \frac{\Delta_h^2\left(F(x) - \frac{1}{2}mx^2\right)}{h^2} \geq 0 \Rightarrow \frac{\Delta_h^2 F(x)}{h^2} \geq m.$$

Similarly $\frac{\Delta_h^2 F(x)}{h^2} \leq M$

Proof of Theorem 28 1. $a_n \cos nx + b_n \sin nx \to 0$ for all x as $n \to \infty \Rightarrow$ $a_n \to 0$ $b_n \to 0$ as $n \to \infty$

2.
$$F(t) = \frac{1}{4}a_0 t^2 - \sum_{n=1}^{\infty}\left(\frac{a_n}{n^2}\cos nt + \frac{b_n}{n^2}\sin nt\right)$$

is uniformly convergent in every finite interval and so $F(t)$ is continuous everywhere.

3. For every given x

$$\frac{1}{2}a_0 + suma_n \cos nx + b_n \sin nx$$

converges to $f(x)$, say therefore $\exists D^2F(x) = f(x)$ therefore $\frac{\Delta_h^2 F(x)}{h^2} \to f(x)$ as $h \to 0$

4.

$$|F(x)| \leq M \Rightarrow \left|\frac{\Delta_h^2 F(x)}{h^2}\right| \leq M$$

for all x in $[-\pi\ \pi]$ and all h sufficiently small.

5.

$$\begin{aligned} \frac{\Delta_h^2 F(x)}{h^2} &= \frac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx)\left(\frac{\sin\frac{1}{2}nh}{\frac{1}{2}nh}\right)^2 \\ &= \frac{1}{2}a_0 + \sum_{n=1}^\infty A_n \cos nx + B_n \sin nx \end{aligned}$$

where $A_n = a_n\left(\frac{\sin\frac{1}{2}nh}{\frac{1}{2}nh}\right)^2 \quad B_n = b_n\left(\frac{\sin\frac{1}{2}nh}{\frac{1}{2}nh}\right)^2$

6. For any fixed h

$$|A_n| = |a_n|\left(\frac{\sin\frac{1}{2}nh}{\frac{1}{2}nh}\right)^2 \leq \frac{4}{h^2}\frac{|A_n|}{n^2}. \quad |B_n| \leq \frac{4}{h^2}\frac{|b_n|}{n^2}$$

therefore $\sum_1^\infty A_n \cos nx + B_n \sin nx$ converges uniformly with respect to x in $[-\pi\ \pi]$.
Therefore

$$A_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\Delta_h^2 F(x)}{h^2} \cos nx\, dx$$

$$B_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\Delta_h^2 F(x)}{h^2} \sin nx\, dx$$

Therefore

$$\lim_{h\to 0} a_m\left(\frac{\sin\frac{1}{2}nh}{\frac{1}{2}nh}\right)^2 = \lim_{h\to 0}\int_{-\pi}^{\pi} \frac{\Delta_h^2 F(x)}{h^2} \sin nx\, dx$$

$\frac{\Delta_h^2 F(x)}{h^2} \to f(x)$ boundedly for $-\pi \leq x \leq \pi$ as $h \to 0$ therefore $a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx\, dx$.
Hence the result.