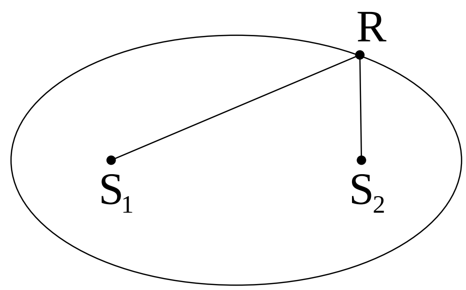
Question

The position vectors of the foci of an ellipse are $\pm\mathbf{c}$ and the length of the major axis is $2a$. Show that the equation of the ellipse is given by

$$a^4 - a^2(\mathbf{r}^2 + \mathbf{c}^2) + (\mathbf{r}\cdot\mathbf{c})^2 = 0.$$

Answer



$|S_1R| + |S_2R| = 2a$
$|\mathbf{r}-\mathbf{c}|^2 + |\mathbf{r}+\mathbf{c}|^2 = 2a$
$|\mathbf{r}-\mathbf{c}|^2 + |\mathbf{r}+\mathbf{c}|^2 + 2|\mathbf{r}-\mathbf{c}||\mathbf{r}+\mathbf{c}| = 4a^2$
Now

$$\begin{aligned} |\mathbf{r}-\mathbf{c}|^2 &= \mathbf{r}\cdot\mathbf{r} - 2\mathbf{r}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{c} \\ |\mathbf{r}+\mathbf{c}|^2 &= \mathbf{r}\cdot\mathbf{r} + 2\mathbf{r}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{c} \end{aligned}$$

So

$$\begin{aligned} |\mathbf{r}-\mathbf{c}||\mathbf{r}+\mathbf{c}| &= 2a^2 - (r^2 + c^2) \\ |\mathbf{r}-\mathbf{c}|^2|\mathbf{r}+\mathbf{c}|^2 &= (r^2 + c^2 - 2rc)(r^2 + c^2 + 2rc) \\ &= (r^2 + c^2)^2 - 4(rc)^2 \end{aligned}$$

So

$$\begin{aligned} (r^2 + c^2)^2 - 4(rc)^2 &= 4a^4 - 4a^2(r^2 + c^2) + (r^2 + c^2)^2 \\ a^4 - a^2(r^2 + c^2) + (rc)^2 &= 0 \end{aligned}$$