Question

For each of the functions given below, determine whether or not $\lim_{x\to 0} f(x)$ exists; if the limit does exist, determine its value wherever possible.

1. $f(x) = \sin(x)\sin(\frac{1}{x})$, for $x \neq 0$;
2. $f(x) = \cos(x)$ for $x \neq 0$, and $f(0) = 2$;
3. $f(x) = [3x + 1]$ (where $[x]$ is the largest integer or floor function);
4. $f(x) = \sin(\sin(\frac{1}{x}))$, for $x \neq 0$;
5. $f(x) = \cos(x)$, if x is a positive rational multiple of π, and $f(x) = 1$ otherwise;
6. $f(x) = \frac{\sin(x)}{|x|}$ for $x \neq 0$;

Answer

1. use the squeeze law. We have that $-1 \leq \sin(\frac{1}{x}) \leq 1$ for all $x \neq 0$, and that $\lim_{x\to 0}\sin(x) = 0$. So, we can bound $f(x)$ below by $-\sin(x)$ and above by $\sin(x)$. Since $\lim_{x\to 0} -\sin(x) = \lim_{x\to 0}\sin(x) = 0$, we have that $\lim_{x\to 0}\sin(x)\sin(\frac{1}{x}) = 0$. [Note that the fact that $f(x)$ is not defined at 0 does not matter, since evaluating $\lim_{x\to 0} f(x)$ depends only on what's happening with $f(x)$ near 0, and not at all on what's happening at 0.]

2. since $\lim_{x\to 0}\cos(x) = 1$, and since $f(x) = \cos(x)$ except at 0, we have that $\lim_{x\to 0} f(x) = \lim_{x\to 0}\cos(x) = 1$. [This is another reflection of the fact that $\lim_{x\to 0} f(x)$ does not care about the value of $f(x)$ at 0, but only on the values of $f(x)$ near 0.]

3. note that $f(x) = 0$ for $-\frac{1}{3} < x \leq 0$, and so $\lim_{x\to 0-} f(x) = 0$. Also, $f(x) = 1$ for $0 < x < \frac{1}{3}$, and so $\lim_{x\to 0+} f(x) = 1$. Since $\lim_{x\to 0+} f(x) \neq \lim_{x\to 0-} f(x)$, we see that $\lim_{x\to 0} f(x)$ does not exist.

4. as $x \to 0+$, we see that $\frac{1}{x} \to \infty$, and so $\sin(\frac{1}{x})$ oscillates between -1 and 1. Hence, as $x \to 0+$, we have that $f(x)$ oscillates between $\sin(-1)$ and $\sin(1)$, and so $\lim_{x\to 0+} f(x)$ does not exist. Hence, $\lim_{x\to 0} f(x)$ does not exist.

5. we apply the squeeze rule, since $\cos(x) \leq f(x) \leq 1$ for all x near 0. Since both $\lim_{x\to 0}\cos(x) = 1$ and $\lim_{x\to 0} 1 = 1$, we have that $\lim_{x\to 0} f(x) = 1$.

6. to evaluate this limit, we recall from calculus that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, and so by the lemma below, we have that $\lim_{x\to 0+} \frac{\sin(x)}{x} = \lim_{x\to 0-} \frac{\sin(x)}{x} = 1$.

   Lemma:

   $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = L$.

   For $x > 0$, we have that $|x| = x$, and so $\lim_{x\to 0+} \frac{\sin(x)}{|x|} = \lim_{x\to 0+} \frac{\sin(x)}{x} = 1$. However, for $x < 0$, we have that $|x| = -x$, and so $\lim_{x\to 0-} \frac{\sin(x)}{|x|} = -\lim_{x\to 0-} \frac{\sin(x)}{x} = -1$. Since $\lim_{x\to 0+} \frac{\sin(x)}{|x|} \neq \lim_{x\to 0-} \frac{\sin(x)}{|x|}$, we see that $\lim_{x\to 0} \frac{\sin(x)}{|x|}$ does not exist.