QUESTION

Using complex contour integration evaluate

(a)

$$\int_0^{2\pi} \frac{d\theta}{1+8\sin^2\theta}$$

(b)

$$\int_0^{2\pi} \frac{d\theta}{3+4i\cos\theta}$$

ANSWER

(a)

$$\begin{aligned} I &= \int_0^{2\pi} \frac{d\theta}{1+8\sin^2\theta} \\ &= \int_{|z|=1} \frac{1}{1+8\left(\frac{1}{2i}\left(z-z^{-1}\right)\right)^2} \frac{dz}{iz} \\ &= \int_{|z|=1} \frac{dz}{iz\left(1-2z^2-2z^{-2}+4\right)} \\ &= \frac{i}{2}\int \frac{zdz}{z^4+1-\frac{5}{2}z^2} \end{aligned}$$

Now consider where the poles of this are.

$\left(z_0-\frac{5}{4}\right)^2 = -1+\frac{25}{16} = \frac{9}{16}, z_0^2 = \frac{5}{4} \pm \frac{3}{4},\ z_0^2 = 2,\ \frac{1}{2},\ z_0 = \pm\sqrt{2},\ \pm\frac{1}{\sqrt{2}}$

There are 4 simple poles, 2 inside the contour.

$\text{Res}\left(\frac{z}{z^4+1-\frac{5}{2}z^2}, z_0\right) = \frac{z_0}{4z_0^3-5z_0} = \frac{1}{4z_0^2-5}$

$\text{Res}\left(\pm\frac{1}{\sqrt{2}}\right) = -\frac{1}{3} \Rightarrow I = 2\pi i \cdot \frac{i}{2} \cdot 2 \cdot \left(-\frac{1}{3}\right) = \frac{2\pi}{3}$

(b)

$$\begin{aligned} I &= \int_0^{2\pi} \frac{d\theta}{3+4i\cos\theta} \\ &= \int_{|z|=1} \frac{dz}{iz\left(3+2i\left(z^{-1}+z\right)\right)} \\ &= -\frac{i}{2}\int \frac{dz}{z^2+1-\frac{3}{2}iz} \end{aligned}$$

Now consider where the poles of this are.

$\left(z - \frac{3}{4}i\right)^2 = -1 - \frac{9}{16} = -\frac{25}{16}$, $z_0 = \left(\frac{3}{4} \pm \frac{5}{4}\right) i$, $z_0 = 2i, -\frac{i}{2}$ There are 2 simple poles, 1 of which is inside the contour.

Res $\left(\frac{1}{z^2+1-\frac{3}{2}iz}, z_0\right) = \frac{1}{2z_0-\frac{3}{2}i}$, Res $\left(-\frac{i}{2}\right) = \frac{2i}{5}$

$$I = 2\pi i \left(-\frac{1}{2}\right) \frac{2i}{5} = \frac{2\pi}{5}$$