QUESTION

Find the residues at each of the singularities of each of the following functions, stating the type of singularity in each case and, if a pole, its order:

$$\text{(i)}\ \frac{1}{z-6i} \quad \text{(ii)}\ \frac{z^2}{z^3-1} \quad \text{(iii)}\ \frac{1}{(z^2+4)^2} \quad \text{(iv)}\ \frac{1}{z}e^{\frac{1}{z^2}}$$

ANSWER

(i) $\frac{1}{z-6i}$: $z = 6i$ is a simple pole, Res $= 1$

(ii) $\frac{z^2}{z^3-1}$: $z = 1,\ e^{\frac{2\pi i}{3}},\ e^{\frac{4\pi i}{3}}$ are simple poles. Res $= \lim_{z\to a_i} \frac{z^2}{3z^2} = \frac{1}{3}$ for each pole by l'Hôpital.

(iii) $\frac{1}{(z^2+4)^2}$, $z = \pm 2i$ are double poles.

$$\begin{aligned} & \text{Res}\left(\frac{1}{(z+2i)^2(z-2i)^2}, 2i\right) \\ = & \lim_{z\to 2i} \frac{d}{dz}\frac{1}{(z+2i)^2} \\ = & \lim_{z\to 2i} -\frac{2}{(z+2i)^3} \\ = & -\frac{i}{32} \end{aligned}$$

$$\text{Res}\left(\frac{1}{(z+2i)^2(z-2i)^2}, -2i\right) = \frac{i}{32}$$

(iv) $\frac{1}{z}e^{\frac{1}{z^2}}$: $z = 0$ is an essential singularity.

$\frac{1}{z}e^{\frac{1}{z^2}} = \frac{1}{z}\left(1 + \frac{1}{z^2} + \frac{1}{2!}\frac{1}{z^4} + \dots\right) \Rightarrow$Res $= 1$