Question
Evaluate the following limits.

1. $\lim_{x\to 2}(1-\cos(\pi x))/\sin^2(\pi x)$;
2. $\lim_{x\to -1}(x^7+1)/(x^3+1)$;
3. $\lim_{x\to 3}(1+\cos(\pi x))/\tan^2(\pi x)$;
4. $\lim_{x\to 1}(1-x+\ln(x))/(1+\cos(\pi x))$;
5. $\lim_{x\to\infty}(\ln(x))^{1/x}$;
6. $\lim_{x\to 2}(x^2+x-6)/(x^2-4)$;
7. $\lim_{x\to 0}(x+\sin(2x))/(x-\sin(2x))$;
8. $\lim_{x\to 0}(e^x-1)/x^2$;
9. $\lim_{x\to 0}(e^x+e^{-x}-x^2-2)/(\sin^2(x)-x^2)$;
10. $\lim_{x\to\infty}\ln(x)/x$;
11. $\lim_{x\to 2}(x^3-x^2-x-2)/(x^3-3x^2+3x-2)$;
12. $\lim_{x\to 1}(x^3-x^2-x+1)/(x^3-2x^2+x)$;

Answer
[Note that for some of these limits, we do not need to use as heavy a piece of machinery as l'Hopital's rule, just some clever simplifying.]

1. since this limit has the indeterminate form $\frac{0}{0}$ (since both $\lim_{x\to 2}(1-\cos(\pi x))=0$ and $\lim_{x\to 2}\sin^2(\pi x)=0$), we may use l'Hopital's rule:

$$\lim_{x\to 2}\frac{1-\cos(\pi x)}{\sin^2(\pi x)}=\lim_{x\to 2}\frac{\pi\sin(\pi x)}{2\pi\sin(\pi x)\cos(\pi x)}=\lim_{x\to 2}\frac{1}{2\cos(\pi x)}=\frac{1}{2}.$$

   (Note that we may also evaluate this limit without l'Hopital's rule, using the trigonometric identity $\sin^2(\theta)+\cos^2(\theta)=1$, as follows:

$$\lim_{x\to 2}\frac{1-\cos(\pi x)}{\sin^2(\pi x)}=\lim_{x\to 2}\frac{1-\cos(\pi x)}{1-\cos^2(\pi x)}=\lim_{x\to 2}\frac{1}{1+\cos(\pi x)}=\frac{1}{2}.\ )$$

2. again, here we have the choice of factoring or using l'Hopital's rule. I feel like factoring:

$$\begin{aligned}\lim_{x\to -1}\frac{x^7+1}{x^3+1} &= \lim_{x\to -1}\frac{(x+1)(x^6-x^5+x^4-x^3+x^2-x+1)}{(x+1)(x^2-x+1)}\\ &= \lim_{x\to -1}\frac{x^6-x^5+x^4-x^3+x^2-x+1}{x^2-x+1}=\frac{7}{3}.\end{aligned}$$

3. write $\tan(z)=\sin(z)/\cos(z)$ and simplify:

$$\begin{aligned}\lim_{x\to 3}\frac{1+\cos(\pi x)}{\tan^2(\pi x)} &= \lim_{x\to 3}\frac{(1+\cos(\pi x))\cos^2(\pi x)}{\sin^2(\pi x)}\\ &= \lim_{x\to 3}\frac{(1+\cos(\pi x))\cos^2(\pi x)}{1-\cos^2(\pi x)}=\lim_{x\to 3}\frac{\cos^2(\pi x)}{1-\cos(\pi x)}=\frac{1}{2}.\end{aligned}$$

4. as this has the indeterminate form $\frac{0}{0}$, and since there seems to be no easy simplification possible, we use l'Hopital's rule:

$$\lim_{x\to 1}\frac{1-x+\ln(x)}{1+\cos(\pi x)}=\lim_{x\to 1}\frac{-1+\frac{1}{x}}{-\pi\sin(\pi x)}.$$

Since this limit still has the indeterminate form $\frac{0}{0}$, we may use l'Hopital's rule again:

$$\lim_{x\to 1}\frac{-1+\frac{1}{x}}{-\pi\sin(\pi x)}=\lim_{x\to 1}\frac{-\frac{1}{x^2}}{-\pi^2\cos(\pi x)}=-\frac{1}{\pi^2}.$$

5. this has the indeterminate form ∞^0, and so we rewrite it:

$$\lim_{x\to\infty}(\ln(x))^{1/x}=\lim_{x\to\infty}\left(e^{\ln(\ln(x))}\right)^{1/x}=e^{\lim_{x\to\infty}\ln(\ln(x))/x}.$$

The exponent has the indeterminate form $\frac{\infty}{\infty}$, and so we may use l'Hopital's rule:

$$\lim_{x\to\infty}\frac{\ln(\ln(x))}{x}=\lim_{x\to\infty}\frac{\frac{1}{\ln(x)}\cdot\frac{1}{x}}{1}=0.$$

Hence, we see that

$$\lim_{x\to\infty}(\ln(x))^{1/x}=e^{\lim_{x\to\infty}\ln(\ln(x))/x}=e^0=1.$$

6. factoring, we see that

$$\lim_{x\to 2}\frac{x^2+x-6}{x^2-4}=\lim_{x\to 2}\frac{(x-2)(x+3)}{(x-2)(x+2)}=\lim_{x\to 2}\frac{x+3}{x+2}=\frac{5}{4}.$$

7. as this limit has the indeterminate form $\frac{0}{0}$, we may use l'Hopital's rule:

$$\lim_{x\to 0}\frac{x+\sin(2x)}{x-\sin(2x)}=\lim_{x\to 0}\frac{1+2\cos(2x)}{1-2\cos(2x)}=\frac{1+2}{1-2}=-3.$$

8. since this limit has the indeterminate form $\frac{\infty}{\infty}$, we may apply l'Hopital's rule:

$$\lim_{x\to 0}\frac{e^x-1}{x^2}=\lim_{x\to 0}\frac{e^x}{2x}=\lim_{x\to 0}\frac{e^x}{2}=\infty.$$

(The second equality follows from applying l'Hopital's rule a second time, which is valid since the limit still has the indeterminate form $\frac{\infty}{\infty}$.)

9. in this limit, though we need to check at each stage, we will apply l'Hopital's rule four times, as the original limit has the indeterminate form $\frac{0}{0}$, and each of the first three applications of l'Hopital's rule results in a limit still in the indeterminate form $\frac{0}{0}$.

$$\begin{aligned}\lim_{x\to 0}\frac{e^x+e^{-x}-x^2-2}{\sin^2(x)-x^2}=\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{2\sin(x)\cos(x)-2x} &= \lim_{x\to 0}\frac{e^x-e^{-x}-2x}{\sin(2x)-2x}\\ &= \lim_{x\to 0}\frac{e^x+e^{-x}-2}{2\cos(2x)-2}\\ &= \lim_{x\to 0}\frac{e^x-e^{-x}}{-4\sin(2x)}\\ &= \lim_{x\to 0}\frac{e^x+e^{-x}}{-8\cos(2x)}=-\frac{1}{4}.\end{aligned}$$

10. this limit has the indeterminate form $\frac{\infty}{\infty}$, and so we apply l'Hopital's rule:

$$\lim_{x\to\infty}\frac{\ln(x)}{x}=\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=0.$$

11. here, we first attempt to evaluate the limit by factoring, a sensible first step for limits of rational functions:

$$\lim_{x\to 2}\frac{x^3-x^2-x-2}{x^3-3x^2+3x-2}=\lim_{x\to 2}\frac{(x-2)(x^2+x+1)}{(x-2)(x^2-x+1)}=\lim_{x\to 2}\frac{x^2+x+1}{x^2-x+1}=\frac{7}{3}.$$

12. again, we first attempt to evaluate the limit by factoring:

$$\lim_{x\to 1}\frac{x^3-x^2-x+1}{x^3-2x^2+x}=\lim_{x\to 1}\frac{(x-1)(x^2-1)}{x(x-1)^2}=\lim_{x\to 1}\frac{x+1}{x}=2.$$