QUESTION

Solve the differential equation

$$y'' + y' - 2y = x^2 + e^x$$

given that $y = 1$ and $y' = 0$ when $x = 0$.

(a) by using particular integral and complementary function,

(b) by using Laplace transforms.

ANSWER

$y'' + y' - 2y = x^2 + e^x, \quad y(0) = 1, \quad y'(0) = 0$

(a) Complementary function: try $y = e^{mx}$

$m^2 + m - 2 = 0 \Rightarrow m_1 = -2, m_2 = 1$

$y_{CF} = C_1e^{-2x} + C_2e^x$

To find a particular integral of $y'' + y' - 2y = x^2$ try $y = Ax^2 + Bx + C$

$2A + (2Ax + B) - 2(ax^2 + Bx + C) = x^2$

$A = -\frac{1}{2} \Rightarrow B = -\frac{1}{2} \Rightarrow C = -\frac{3}{4}$

To find a particular integral of $y'' + y' - 2y = e^x$ try $y = p(x)e^x$

$(D + 2)(D - 1)p(x)e^x = e^x \Rightarrow (D + 3)Dp(x) = 1$

$p(x) = \frac{1}{D}\frac{1}{D+3}1 = \frac{1}{D}\frac{1}{3} = \frac{1}{3}x$

Hence the general solution is

$$\begin{aligned}
y &= C_1e^{-2x} + C_2e^x - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4} + \frac{1}{3}xe^x \\
y(0) &= C_1 + C_2 - \frac{3}{4} = 1 \Rightarrow C_2 = -C_1 + \frac{7}{4} \\
y'(0) &= -2C_1 + C_2 - \frac{1}{2} + \frac{1}{3} = 0 \\
&\Rightarrow 3C_1 = -\frac{6}{12} + \frac{4}{12} + \frac{21}{12} = \frac{19}{21} \\
&\Rightarrow C_1 = \frac{19}{36} \\
&\Rightarrow C_2 = -\frac{19}{36} + \frac{63}{36} = \frac{11}{9}
\end{aligned}$$

(b)

$$\left(s^2Y(s) - sy(0) - y'(o)\right) + (sY(s) - y(0)) - 2Y(S) = \frac{2}{s^3} + \frac{1}{s-1}$$

$$(s^2+s-2)Y(s) = \frac{2}{s^3} + \frac{1}{s-1} + s + 1$$

$$Y(s) = \frac{2}{s^3(s+2)(s-1)} + \frac{1}{(s+2)(s-1)^2} + \frac{s+1}{(s+2)(s-1)}$$

Res $\left(\frac{2e^{sx}}{s^3(s+2)(s-1)}, 0\right)$ (a triple pole)=coefficient of s^{-1} in

$$-\frac{e^{sx}}{s^3\left(1+\frac{s}{2}\right)(1-s)} = -\frac{1}{s^3}\left(1+sx+\frac{s^2x^2}{2!}+\dots\right)\left(1-\frac{s}{2}+\frac{s^2}{4}+\dots\right)$$

$$\left(1+s+s^2+\dots\right) - \frac{1}{s^3}\left[1+\left(x-\frac{1}{2}+1\right)s+\left(\frac{x^2}{2}+\frac{1}{4}+1-\frac{x}{2}+x-\frac{1}{2}\right)s^2+\dots\right]$$

Res $= -\frac{x^2}{2} - \frac{x}{2} - \frac{3}{4}$

Res $(Y(s)e^{sx}, -2)$ (a simple pole)

$$\begin{aligned}
&= e^{-2x}\left(\frac{2}{(-2)^3(-2-1)} + \frac{1}{(-2-1)^2} + \frac{-2+1}{(-2-1)}\right) \\
&= e^{-2x}\left(\frac{1}{\sqrt{2}} + \frac{1}{9} + \frac{1}{3}\right) = \frac{19}{36}e^{-2x}
\end{aligned}$$

$$\begin{aligned}
\text{Res}\,(Y(s)e^{sx}, 1) &= e^x\left(\frac{2}{1(1+2)} + \frac{1+1}{1+2}\right) + \lim_{s\to 1}\frac{d}{ds}\frac{e^s x}{s+2} \\
&= \frac{4}{3}e^x + \lim_{x\to 1}\left(\frac{xe^{sx}}{s+2} - \frac{e^{sx}}{(s+2)^2}\right) \\
&= \left(\frac{4}{3} + \frac{x}{3} - \frac{1}{9}\right)e^x \\
&= \frac{x}{3} + \frac{11}{9}
\end{aligned}$$

$$y(x) = \frac{19}{36}e^{-2x} + \left(\frac{x}{3} + \frac{11}{9}\right)e^x - \frac{x^2}{2} - \frac{x}{2} - \frac{3}{4}$$

as in (a).