Exam Question
Topic: DiffInt
The function f satisfies the equation

$$f(x) = \sqrt{3} + \int_0^x [1 + (f(t))^2]\, dt.$$

By differentiating the integral, find a differential equation for $f(x)$. Hence find the function $f(x)$.

Solution

$$f(x)]\sqrt{3} + \int_0^x [1 + (f(t))^2]\, dt$$

Differentiating with respect to x gives

$$\frac{df}{dx} = 1 + (f(x))^2 \quad \text{so} \quad \frac{1}{1 + f^2}\frac{df}{dx} = 1.$$

This gives $\tan^{-1} f = x + c$, so $f(x) = \tan(x + c)$.
Now $f(0) = \tan c = \sqrt{3}$ so $c = \pi/3$. Hence $f(x) = \tan\left(x + \frac{\pi}{3}\right)$.