Question

In this question, A is a subset of $\mathbf{R}$. Define $A^- = \{-a \mid a \in A\}$. Show that each of the following holds.

1. if $\sup(A)$ exists, then $\inf(A^-)$ exists and $\inf(A^-) = -\sup(A)$;
2. if $\inf(A)$ exists, then $\sup(A^-)$ exists and $\sup(A^-) = -\inf(A)$.

Answer

1. Since $\sup(A)$ exists, the set A is bounded above. Let u be any upper bound for A, so that $a \leq u$ for all $a \in A$. Multiplying through by -1, this becomes $-a \geq -u$ for all $a \in A$. Since $-a$ ranges over all of A^- as a ranges over A, this yields that $-u$ is a lower bound for A^-, and so $\inf(A^-)$ exists. In particular, taking $u = \sup(A)$, we have that $-\sup(A)$ is a lower bound for A^-.

   To see that there is no lower bound for A^- that is greater than $-\sup(A)$, note that t is a lower bound for A^- if and only if $-t$ is an upper bound for A. Therefore, a lower bound for A^- greater than $-\sup(A)$ exists if and only if an upper bound for A less than $\sup(A)$ exists, but by the definition of supremum no such upper bound can exist. Hence, $-\sup(A)$ is the greatest lower bound for A^-, or in other words, $-\sup(A) = \inf(A^-)$, as desired.

2. Since $\inf(A)$ exists, the set A is bounded below. Let t be any lower bound for A, so that $a \geq t$ for all $a \in A$. Multiplying through by -1, this becomes $-a \leq -t$ for all $a \in A$. Since $-a$ ranges over all of A^- as a ranges over A, this yields that $-t$ is an upper bound for A^-, and so $\sup(A^-)$ exists. In particular, taking $t = \inf(A)$, we have that $-\inf(A)$ is an upper bound for A^-.

   To see that there is no upper bound for A^- that is less than $-\inf(A)$, note that u is an upper bound for A^- if and only if $-u$ is a lower bound for A. Therefore, an upper bound for A^- less than $-\inf(A)$ exists if and only if a lower bound for A greater than $\inf(A)$ exists, but by the definition of infimum no such lower bound can exist. Hence, $-\inf(A)$ is the least upper bound for A^-, or in other words, $-\inf(A) = \sup(A^-)$, as desired.