Exam Question
Topic: CriticalPoints
Find and classify the critical points of the function

$$f(x,y) = y(x-2)^2 + 2x^2 + y^2 - 8x - 12y.$$

Calculate the value of the function at each of the critical points.

Solution

$$\begin{aligned} f(x,y) &= y(x-2)^2 + 2x^2 + y^2 - 8x - 12y \\ f_x &= 2y(x-2) + 4x - 8 = 2(y+2)(x-2) \\ f_y &= (x-2)^2 + 2y - 12. \\ \text{So } f_x &= 0 \Rightarrow y = -2 \text{ or } x = 2. \end{aligned}$$

When $x = 2, f_y = 0 \Rightarrow 2y - 12 = 0;\ y = 6.$
When $y = -2, f_y = 0 \Rightarrow (x-2)^2 - 4 - 12 = 0,$
giving $(x-2)^2 = 16;\ x = 2 \pm 4;\ x = 6$ or $x = -2.$
So the critical points are $(2,6); (6,-2); (-2,-2).$
The second partial derivatives are given by

$$f_{xx} = 2y + 4;\ f_{yy} = 2;\ f_{xy} = 2(x-2).$$

Classifying the critical points gives

	$2(x-2)$	$2y+4$	2	D	
$(2,6)$	0	16	2	$-32 < 0$	MIN
$(6,-2)$	8	0	2	$64 > 0$	SADDLE
$(-2,-2)$	-8	0	2	$64 > 0$	SADDLE