Question

(i) Find the parabola of the form $Ax + By^2 + cy + D = 0$ through the three points $P_1(-1,0)$, $P_2(2,1)$, $P_3(1,-1)$. Find the vertex and the axis of the parabola and sketch.

(ii) Find a point on intersection of the parabolas

$$r = \frac{1}{1-\cos\theta} \qquad r = \frac{3}{1+\cos\theta}$$

and find the angle between the tangents to these curves at this point.

Answer

(i)

$$\begin{aligned} Ax + By^2 + Cy + D &= 0 \\ \text{So} \qquad A + D &= 0 \\ 2A + B + C + D &= 0 \\ A + B - C + D &= 0 \end{aligned}$$

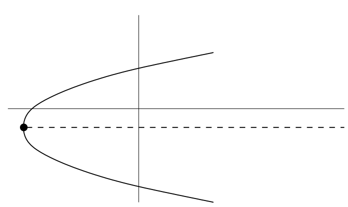
Choose $A = D = 2$ then $B + C = -6$, $B - C = -4$, so $B = -5$, $C = -1$

So the parabola is

$$2x - 5y^2 - y + 2 = 0$$

or

$$\left(y + \frac{1}{10}\right)^2 = \frac{2}{5}\left(x + \frac{41}{40}\right)$$



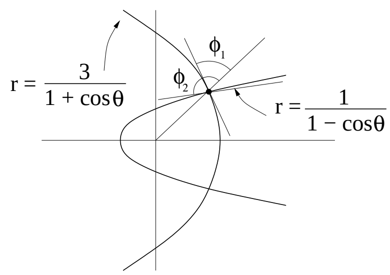
Vertex $= \left(-\frac{41}{40}, -\frac{1}{10}\right)$ axis $y = -\frac{1}{10}$

(ii) The parabolas intersect where $\dfrac{1}{1-\cos\theta} = \dfrac{3}{1+\cos\theta}$

giving $\cos\theta = \dfrac{1}{2} \quad \theta = \pm\dfrac{\pi}{3}$.

When $\cos\theta = \frac{1}{2}$, $r = 2$

So points of intersection are $\left(2, \pm\dfrac{\pi}{3}\right)$



For $r = \dfrac{1}{1-\cos\theta} \quad \dfrac{dr}{d\theta} = -\dfrac{\sin\theta}{(1-\cos\theta)^2}$

$\dfrac{1}{r}\dfrac{dr}{d\theta} = -\dfrac{\sin\theta}{1-\cos\theta} = -\sqrt{3} \quad \tan\phi_2 = -\dfrac{1}{\sqrt{3}}\phi_2 = 150^\circ$

For $r = \dfrac{3}{1+\cos\theta} \quad \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{\sin\theta}{1+\cos\theta} \quad \tan\phi_1 = \sqrt{3} \quad \phi_1 = 60^\circ$

So the angle between the tangents is 90°